HAMILTONIAN ANALYSIS OF POINCARÉ GAUGE THEORY: HIGHER SPIN MODES

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Abstract

We examine several higher spin modes of the Poincaré gauge theory (PGT) of gravity using the Hamiltonian analysis. The appearance of certain undesirable effects due to non-linear constraints in the Hamiltonian analysis are used as a test. We find that the phenomena of field activation and constraint bifurcation both exist in the pure spin 1 and the pure spin 2 modes. The coupled spin-0^- and spin-2^- modes also fail our test due to the appearance of constraint bifurcation. The "promising" case in the linearized theory of PGT given by Kuhfuss and Nitsch[1] likewise does not pass. From this analysis of these specific PGT modes we conclude that an examination of such nonlinear constraint effects shows great promise as a strong test for this and other alternate theories of gravity.

1 Introduction

Many alternative theories of gravity have been proposed to replace general relativity (GR); experimental and observational results are applied to examine the predictions of each alternative and to eliminate the unsatisfactory ones[2]. Certain gauge theories of gravity, with their many ranges of parameters, not only pass all the experimental and observational tests but also agree

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with GR to Post-Newtonian order[3]. Consequently no discriminating experimental and observational test can presently distinguish them from GR. Thus they are observationally equally viable. Spontaneously many people paid attention to another criteria — theoretical tests. Essentially, besides self-consistency, theoretical tests judge if a theory obeys commonly accepted physical requirements, such as "no tachyons" (faster than light signals) and "no ghosts" (negative energy waves). Moreover a viable theory should have an appropriate mathematical structure. This includes a well-posed initial value problem: the basic requirement is the Cauchy-Kovalevska theorem; beyond that, the propagation of dynamic modes should be described by hyperbolic evolution equations with well-behaved characteristics.

Although the metric-affine gauge theory of gravity (MAG)[4] is more general (it has curvature, torsion and non-metricity), we here apply our Hamiltonian based analysis mainly to the Poincaré gauge theory (PGT), an important sub-theory (with only curvature and torsion, i.e., vanishing nonmetricity) of the MAG. One reason for this choice is physical. In the foreseeable future the effect of nonmetricity is thought to be extremely hard to measure as it is expected to only survive at the Planck energy scale. On the other hand it is believed that the effect of torsion has the possibility to be detected from some strong astronomical sources. However our primary motivation for restricting our considerations to the PGT is that it is simpler. Before examining nonlinear effects in the MAG, we really need more experience. Also there have been many more studies of the behavior of the PGT. These works provide a support for our deeper investigation.

Pioneered by Sezgin and Nieuwenhuizen[5], people examined the PGT based on the "spin-projection operator formalism"[6] and the weak field approximation with positive energy and mass[7]. They put restrictions on the ranges of the PGT parameters and gave tables of "viable" PGT parameter ranges with well-behaved propagating modes. The positive energy test (PET), which excludes the cases permitting a non-trivial solution that has non-positive energy, was proposed as an effective theoretical test for gravitational theories and applied to the PGT[8,9]. Parameter restrictions leading to the interesting purely curvature or torsion modes were identified[10]. On the other hand, Cauchy-Kovalevska and hyperbolicity conditions were imposed on the PGT to study the initial value problems[11]; a shock wave analysis was applied to find the characteristics of propagating modes[12]. However, it should be noted that nonlinear effects were essentially not probed in most of these investigations. Almost all of the above examinations were directly or effectively working on the linearization of the PGT, except for the PET.

The PET has been applied to the PGT and gave some good results with only simple ansatz which are used to find "bad" solutions[13]. The nice feature is that the PET considers the full nonlinear theory and thus is not subject to such doubts that a result of a test derived from the linearized theory suffers. However, the test can be applied only if the total energy is well defined, hence it applies only to the asymptotically flat or constant negative curvature configurations. Actually, in practice the PET has not given many new results beyond those found from the linearized theory tests, simply because it is not so easy to apply in general. This limitation provided further motivation for our nonlinear Hamiltonian investigation.

An early warning of the difficulties (due to nonlinearity) was raised by Kopczyński[14]. He noted that the one parameter teleparallel theory (OPTP, a.k.a. NGR)[15] had predictability problems. Further analysis[16] revealed that the problem was not generic, it occurred only for a special class of solutions. Cheng *et al* using a Hamiltonian approach verified this property[17] and identified the troubles with the phenomenon of *constraint bifurcation*: the chain of constraints could "bifurcate" depending on the values of the fields, so that the number and/or type of

constraints would depend on the values of the phase space variables. Since the OPTP is a sub-theory (with curvature vanishing) of the PGT, it was expected that the PGT as well as the MAG and other alternate geometric gravity theories were also vulnerable to this difficulty. Besides constraint bifurcation, it is likely that the gauge theories of gravity will encounter the nonlinear phenomenon of turning constraints into equations of motion as in the analysis by Velo and Zwanzinger[18] of higher-spin fields. Also within Einstein's theory of GR with sources included, the inherent non-linearity of the gravitational coupling to sources has long been known to have peculiar effects, including the activation of non-dynamic source degrees of freedom (see, in particular, Ref. 19 and §5.2 in Ref. 20). These interesting considerations motivated our investigation which is reported in this paper.

Our main tool for studying these nonlinear effects is the Hamiltonian analysis using the Dirac constraint algorithm in the full nonlinear PGT. The Hamiltonian analysis reveals whether a theory is consistent. It systematically determines the constraints, the gauges, and the degrees of freedom for each consistent theory. The analysis also prescribes methods for reducing the phase space of the theory to one containing only the degrees of freedom[20]. Moreover it is the most straightforward way to find the apparatus of the initial value scheme for a given theory — the proper data, the constraints, and the evolution equations.

We base our test of the viability of a particular case of the PGT (with certain parameter choices) by checking if the whole structure of the theory stays the same before and after linearization. The change of the structure of the theory can be measured by studying the nonlinearity in the Poisson matrix of the constraints. In practice this was not so complicated, for in all the cases we have examined it was sufficient to consider the Poisson matrix of the *primary* constraints. In Ref. 21 we analyzed two modes of PGT with propagating spin-0 torsion. In this paper we continue our examination of the PGT considering now the simple spin-1 modes, simple spin-2 modes, coupled spin-0^- and spin-2^- modes, and the modes of the preferred parameter choice identified by Kuhfuss and Nitsch[1]. We find that two phenomena, i.e., *constraint bifurcation* and *field activation*, could happen in all of these cases[22]. These phenomena prevent these cases from being viable.

This paper is organized as follows. In section 2 we review the basic elements of the PGT and introduce its Lagrangian and field equations. In section 3 we use the Dirac theory for constrained Hamiltonian systems in the neat "if" constraint formulation developed by Blagojević and Nikolić[23,24]. The primary constraints, including ten "sure" primary constraints and thirty so-called "primary if-constraints", are given. The total Hamiltonian density, including the canonical Hamiltonian density and all possible primary constraints, is presented. We summarize the results and the viable parameter combinations of the linearized PGT in section 4. The viable parameter combinations in this section are a guide to search for the viable modes of the full PGT. In section 5 we study the viability of the PGT of the simple spin-1 modes, simple spin-2 modes, coupled spin-0^- and spin-2^- modes, and give an argument to explain the failure of the parameter choice by Kuhfuss and Nitsch. In the final section we present our conclusions.

Throughout the paper our conventions are basically the same as Hehl's in Ref. 25. We have made a few adjustments to accommodate the translation of the Hamiltonian "if" constraint formalism to these conventions. The Latin indices are coordinate (holonomic) indices, whereas the Greek indices are orthonormal frame (an-holonomic) indices. The first letters of both alphabets ($a, b, c, \ldots;\ \alpha, \beta, \gamma, \ldots$) run over 1, 2, 3, whereas the later ones run over 0, 1, 2, 3. Furthermore, $\eta_{\mu\nu}$ =diag$(-,\ +,\ +,\ +)$; $\epsilon^{\mu\nu\gamma\delta}$ is the completely antisymmetric tensor with $\epsilon^{\hat{0}\hat{1}\hat{2}\hat{3}} = -1$. The meaning of a bar over a Greek index is adopted from Ref. 24.

2 Poincaré Gauge Theory of Gravitation

In the PGT there are two sets of gauge potentials, the orthonormal frame field (tetrads) $e_i{}^\mu$ and the metric-compatible connection $\Gamma_{i\mu}{}^\nu$, which are associated with the translation and the Lorentz subgroups of the Poincaré gauge group, respectively. The associated field strengths are the torsion

$$T_{ij}{}^\mu = 2(\partial_{[i} e_{j]}{}^\mu + \Gamma_{[i|\nu}{}^\mu e_{|j]}{}^\nu), \tag{1}$$

and the curvature

$$R_{ij\mu}{}^\nu = 2(\partial_{[i} \Gamma_{j]\mu}{}^\nu + \Gamma_{[i|\sigma}{}^\nu \Gamma_{|j]\mu}{}^\sigma), \tag{2}$$

which satisfy the Bianchi identities

$$\nabla_{[i} T_{jk]}{}^\mu \equiv R_{[ijk]}{}^\mu, \tag{3}$$
$$\nabla_{[i} R_{jk]}{}^{\mu\nu} \equiv 0. \tag{4}$$

The conventional form of the action, which is invariant under the Poincaré gauge group, has the form

$$S = \int d^4x e(L_M + L_G), \tag{5}$$

where L_M stands for the matter Lagrangian density (which determines the energy-momentum and spin source currents), L_G denotes the gravitational Lagrangian density, and $e = det(e_i{}^\mu)$. In this paper we are concerned with the gravitational propagating modes, hence we omit the matter Lagrangian density, so L_G is considered as the source-free total Lagrangian. Varying with respect to the potentials then gives the (vacuum) field equations,

$$\nabla_j H_\mu{}^{ij} - \varepsilon_\mu{}^i = 0, \tag{6}$$
$$\nabla_j H_{\mu\nu}{}^{ij} - \varepsilon_{\mu\nu}{}^i = 0, \tag{7}$$

with the field momenta

$$H_\mu{}^{ij} := \frac{\partial e L_G}{\partial \partial_j e_i{}^\mu} = 2\frac{\partial e L_G}{\partial T_{ji}{}^\mu}, \tag{8}$$
$$H_{\mu\nu}{}^{ij} := \frac{\partial e L_G}{\partial \partial_j \Gamma_i{}^{\mu\nu}} = 2\frac{\partial e L_G}{\partial R_{ji}{}^{\mu\nu}}, \tag{9}$$

and

$$\varepsilon_\mu{}^i := e^i{}_\mu e L_G - T_{\mu j}{}^\nu H_\nu{}^{ji} - R_{\mu j}{}^{\nu\sigma} H_{\nu\sigma}{}^{ji}, \tag{10}$$
$$\varepsilon_{\mu\nu}{}^i := H_{[\nu\mu]}{}^i. \tag{11}$$

The Lagrangian is chosen (as usual) to be at most of quadratic order in the field strengths, then the field momenta are linear in the field strengths:

$$H_\mu{}^{ij} = \frac{e}{l^2} \sum_{k=1}^{3} a_k \overset{(k)}{T}{}^{ji}{}_\mu, \tag{12}$$
$$H_{\mu\nu}{}^{ij} = -\frac{a_0 e}{l^2} e^i{}_{[\mu} e^j{}_{\nu]} + \frac{e}{\kappa} \sum_{k=1}^{6} b_k \overset{(k)}{R}{}^{ji}{}_{\mu\nu}, \tag{13}$$

the three $\overset{(k)}{T}{}^{ji}{}_{\mu}$ and the six $\overset{(k)}{R}{}^{ji}{}_{\mu\nu}$ are the algebraically irreducible parts of the torsion and the curvature, respectively. The reciprocal frames $e^i{}_\mu$ and $e_i{}^\mu$ satisfy $e^i{}_\mu e_i{}^\nu = \delta_\mu{}^\nu$ and $e^i{}_\mu e_j{}^\mu = \delta_j{}^i$; the coordinate metric is defined by $g_{ij} = e_i{}^\mu e_j{}^\nu \eta_{\mu\nu}$. l and κ are constants; the a_k and b_k are free coupling parameters. Due to the Bach-Lanczos identity only five of the six b_k's are independent. a_0 is the coupling parameter of the scalar curvature $R := R_{\mu\nu}{}^{\nu\mu}$. For the Hamiltonian formulation we associate the *canonical momenta* with certain components of the covariant field momenta:

$$\pi^i{}_\mu \equiv H_\mu{}^{i0}, \quad \pi^i{}_{\mu\nu} \equiv H_{\mu\nu}{}^{i0}. \tag{14}$$

3 Total Hamiltonian and Primary If-constrains

The Hamiltonian analysis is our main tool for studying the PGT. We present here the primary constraints and the total Hamiltonian density in terms of the decomposition of the canonical variables. As indicated in Table 1, in addition to certain sure constraints associated with the basic local Poincaré gauge symmetry of spacetime, for each of the kinetic critical parameter combinations which vanishes there is an extra degeneracy. Such extra degeneracies are associated with *extra* primary constraints which can lead to secondaries etc. Essentially we follow the techniques developed by Blagojević and Nikolić, who investigated systematically all of these possibilities of the critical parameter combinations[23,24]. They adopted the *if-constraint* technique which avoids getting stuck in unnecessary detail and still identifies all possible constraints.

The sure primary constraints are

$$\pi^0{}_\mu \approx 0, \quad \pi^0{}_{\mu\nu} \approx 0. \tag{15}$$

These constraints reflect the fact that the torsion and the curvature are defined as the antisymmetric derivatives of $e_i{}^\mu$ and $\Gamma_i{}^{\mu\nu}$; they do not involve the "velocities" $\dot{e}_0{}^\mu$ and $\dot{\Gamma}_0{}^{\mu\nu}$. One will obtain further primary "if"-constraints if the Lagrangian density is singular with respect to any parts of the remaining "velocities", $\dot{e}_a{}^\mu$, and $\dot{\Gamma}_a{}^{\mu\nu}$. The total Hamiltonian density is formally of the form

$$\mathcal{H}_{\mathrm{T}} = \mathcal{H}_{\mathrm{can}} + u_0{}^\mu \pi^0{}_\mu + \frac{1}{2} u_0{}^{\mu\nu} \pi^0{}_{\mu\nu} + u^A \phi_A, \tag{16}$$

where the ϕ_A are the primary "if" constraints and the u's denote the associated *Lagrange multipliers*; $\mathcal{H}_{\mathrm{can}}$ stands for the canonical Hamiltonian density which will be specified below. The primary constraints (15) are first-class, i.e., $\pi^0{}_\mu$ and $\pi^0{}_{\mu\nu}$ are unphysical variables. By using the Hamilton equation of motion we can infer that the multipliers $u_0{}^\mu$ and $u_0{}^{\mu\nu}$ are indeed equal to $\dot{e}_0{}^\mu$ and $\dot{\Gamma}_0{}^{\mu\nu}$. They are dynamically undetermined pure gauge multipliers.

In the PGT we consider a nine-parameter Lagrangian which is of the $R + T^2 + R^2$ type. It is well-known that the "most dynamical" case of the theory, when the parameter combinations are not critical, is of no physical importance due to the presence of ghosts and tachyons[7], one will expect there are "*primary if-constraints*"(PIC). Sec. 3.2 is devoted to the subject. Here we outline the procedure as follows: The first step is to find all the values of parameters which diminish the rank of the Hessian matrices $\partial^2\mathcal{L}/\partial\dot{e}_a{}^\mu\partial\dot{e}_b{}^\nu$ and $\partial^2\mathcal{L}/\partial\dot{\Gamma}_a{}^{\mu\nu}\partial\dot{\Gamma}_b{}^{\tau\sigma}$ — the critical values of the parameters. Such values of parameters result in some torsion modes being frozen. We will show all of the possible primary constraints, which appear when the parameters take on the critical values. They will be written in the form of "if"-constraints, which automatically drop out from the theory when the corresponding critical values of the parameters are not

fulfilled. Finally, we will find the expression for the super-Hamiltonian $\mathcal{H}_\perp$, which is valid for all values of the parameters.

3.1 Total Hamiltonian

Before we proceed to obtain the explicit form of the canonical Hamiltonian density and ϕ_A, it is useful to define the decomposition of the related variables and functions. The components of the unit normal $\mathbf{n}$ to the x^0 =constant hypersurface, with respect to the orthonormal frame, are given by

$$n_\mu := \frac{-e^0{}_\mu}{\sqrt{-g^{00}}}. \tag{17}$$

A vector, e.g., V_μ, can be decomposed into the orthogonal and parallel components with respect to the orthonormal frame indices:

$$\begin{aligned} V_\mu &= -V_\perp n_\mu + V_{\overline{\mu}}, && (18)\\ V_\perp &\equiv V_\mu n^\mu, && (19)\\ V_{\overline{\mu}} &\equiv V_\nu(\delta_\mu{}^\nu + n_\mu n^\nu). && (20)\end{aligned}$$

One can extend the decomposition to any tensors with orthonormal frame indices. The lapse and shift functions can be written as

$$\begin{aligned} N &\equiv \frac{1}{\sqrt{-g^{00}}} = -n_\mu e_0{}^\mu, && (21)\\ N^a &\equiv -\frac{g^{0a}}{g^{00}} = e_0{}^\mu e_{\overline{\mu}}{}^a, && (22)\end{aligned}$$

and $e = -NJ$, where $(-J)$ is the determinant of the 3-metric.

To construct the canonical Hamiltonian density, we decompose the torsion and the curvature tensor as

$$\begin{aligned} T_{\nu\sigma}{}^\mu &= 2n_{[\nu}T_{\overline{\sigma}]\perp}{}^\mu + T_{\overline{\nu\sigma}}{}^\mu, && (23)\\ R_{\tau\sigma}{}^{\mu\nu} &= 2n_{[\tau}R_{\overline{\sigma}]\perp}{}^{\mu\nu} + R_{\overline{\tau\sigma}}{}^{\mu\nu}, && (24)\end{aligned}$$

so that $T_{\overline{\nu\sigma}}{}^\mu$ and $R_{\overline{\tau\sigma}}{}^{\mu\nu}$ are independent of velocities. Defining the convenient "parallel" canonical momenta

$$\pi^{\overline{\sigma}}{}_\mu \equiv e_a{}^\sigma \pi^a{}_\mu, \quad \pi^{\overline{\sigma}}{}_{\mu\nu} \equiv e_a{}^\sigma \pi^a{}_{\mu\nu}, \tag{25}$$

which satisfy $\pi^{\overline{\sigma}}{}_\mu n_\sigma = 0$, $\pi^{\overline{\sigma}}{}_{\mu\nu} n_\sigma = 0$, we obtain

$$\pi^{\overline{\sigma}}{}_\mu \equiv e_a{}^\sigma \frac{\partial e\mathcal{L}}{\partial \dot{e}_a{}^\mu} = J\frac{\partial \mathcal{L}}{\partial T_{\overline{\sigma}\perp}{}^\mu}, \quad \pi^{\overline{\sigma}}{}_{\mu\nu} \equiv e_a{}^\sigma \frac{\partial e\mathcal{L}}{\partial \dot{\Gamma}_a{}^{\mu\nu}} = J\frac{\partial \mathcal{L}}{\partial R_{\overline{\sigma}\perp}{}^{\mu\nu}}. \tag{26}$$

The canonical Hamiltonian density,

$$\mathcal{H}_{\rm can} = \pi^a{}_\mu \dot{e}_a{}^\mu + \frac{1}{2}\pi^a{}_{\mu\nu}\dot{\Gamma}_a{}^{\mu\nu} - e\mathcal{L}, \tag{27}$$

can be rewritten in the so-called Dirac-ADM form[26,27,28],

$$\mathcal{H}_{\rm can} = N\mathcal{H}_\perp + N^a\mathcal{H}_a + \frac{1}{2}\Gamma_0{}^{\mu\nu}\mathcal{H}_{\mu\nu} + \partial_a\mathcal{D}^a, \tag{28}$$

which is linear in N and N^a. The other quantities are given by[24]

$$\begin{aligned}
\mathcal{H}_\perp &= \pi^{\overline{\sigma}}{}_\mu T_{\perp\overline{\sigma}}{}^\mu + \frac{1}{2}\pi^{\overline{\sigma}}{}_{\mu\nu} R_{\perp\overline{\sigma}}{}^{\mu\nu} + J\mathcal{L} - n^\mu \nabla_a \pi^a{}_\mu, &(29)\\
\mathcal{H}_a &= \pi^b{}_\mu T_{ab}{}^\mu + \frac{1}{2}\pi^b{}_{\mu\nu} R_{ab}{}^{\mu\nu} - e_a{}^\mu \nabla_b \pi^b{}_\mu, &(30)\\
\mathcal{H}_{\mu\nu} &= \pi^a{}_\mu e_{a\nu} - \pi^a{}_\nu e_{a\mu} - \nabla_a \pi^a{}_{\mu\nu}, &(31)\\
\mathcal{D}^a &= \pi^a{}_\mu e_0{}^\mu + \frac{1}{2}\pi^a{}_{\mu\nu} \Gamma_0{}^{\mu\nu}. &(32)
\end{aligned}$$

With the help of Eq. (26) we can rewrite the expression in the parentheses in Eq. (29) as a function of $T_{\overline{\nu\sigma}}{}^\mu$, $\pi^{\overline{\sigma}}{}_\mu$, $R_{\overline{\tau\sigma}}{}^{\mu\nu}$ and $\pi^{\overline{\sigma}}{}_{\mu\nu}$. Thus it is independent of the unphysical variables and represents the only dynamical part of the canonical Hamiltonian density. By combining (16) and (28) the total Hamiltonian is

$$\begin{aligned}
\mathcal{H}_{\mathrm{T}} &= N\mathcal{H}_\perp + N^a\mathcal{H}_a + \frac{1}{2}\Gamma_0{}^{\mu\nu}\mathcal{H}_{\mu\nu} + \partial_a \mathcal{D}^a \\
&\quad + u_0{}^\mu \pi^0{}_\mu + \frac{1}{2} u_0{}^{\mu\nu}\pi^0{}_{\mu\nu} + u^A \phi_A. &(33)
\end{aligned}$$

According to Dirac's general arguments, there have to be (at least) ten first-class constraints in the PGT, as it is invariant under the ten-parameter local Poincaré group. By utilizing the total Hamiltonian density (33) in the consistency conditions for the sure primary constraints (15) we obtain the *sure secondary constraints* (SC)

$$\mathcal{H}_\perp \approx 0, \quad \mathcal{H}_a \approx 0, \quad \mathcal{H}_{\mu\nu} \approx 0. \tag{34}$$

$\mathcal{H}_\perp$ and $\mathcal{H}_a$ represent the generators of the orthogonal and parallel x^0 =const hypersurface deformations[29,30,31]. And $\mathcal{H}_{\mu\nu}$ represents the generators of the local Lorentz transformations[32,33,34]. They are all first-class constraints, so that their consistency conditions are trivially satisfied[35]. Only the super-Hamiltonian $\mathcal{H}_\perp$ is involved in the dynamical evolution, the super-momenta $\mathcal{H}_a$ and Lorentz rotation parts $\mathcal{H}_{\mu\nu}$ are kinematic generators, consequently we concentrate on $\mathcal{H}_\perp$ when consistency conditions are calculated.

3.2 Primary if-constraints and super-Hamiltonian

It is necessary to understand the relation between the canonical momenta and the velocities before one can make $\mathcal{H}_\perp$ more apparent. The torsion momenta $\pi_{\overline{\nu}\mu}$ can be decomposed into four algebraically irreducible parts:

$$\begin{aligned}
\pi_{\overline{\mu}\nu} &= -n_\nu \pi_{\overline{\mu}\perp} + \pi_{\overline{\mu\nu}} \\
&= -n_\nu \pi_{\overline{\mu}\perp} + \overset{\wedge}{\pi}_{\overline{\mu\nu}} + \overset{\sim}{\pi}_{\overline{\mu\nu}} + \frac{1}{3}\eta_{\overline{\mu\nu}}\pi, &(35)
\end{aligned}$$

where the notations refer respectively to the antisymmetric, symmetric-traceless and the trace parts. Manipulating the definition of the torsion momenta (14), the following relations between the different parts of the canonical momenta and the corresponding parts of the velocities $T_{\perp\overline{\mu}\nu}$ are found:

$$\phi_{\overline{\mu}\perp} \equiv \frac{\pi_{\overline{\mu}\perp}}{J} + \frac{1}{3l^2}(a_1 - a_2)\overset{\rightharpoonup}{T}_{\overline{\mu}} = \frac{1}{3l^2}(2a_1 + a_2)T_{\perp\overline{\mu}\perp}, \tag{36}$$

$$\overset{\wedge}{\phi}_{\overline{\mu\nu}} \equiv \frac{\overset{\wedge}{\pi}_{\overline{\mu\nu}}}{J} + \frac{1}{3l^2}(a_1 - a_3)T_{\overline{\mu\nu}\perp} = \frac{1}{3l^2}(a_1 + 2a_3)T_{\perp[\overline{\mu\nu}]}, \tag{37}$$

$$\overset{\sim}{\phi}_{\overline{\mu\nu}} \equiv \frac{\overset{\sim}{\pi}_{\overline{\mu\nu}}}{J} = \frac{a_1}{l^2}T_{\perp\langle\overline{\mu\nu}\rangle}, \tag{38}$$

$$\phi \equiv \frac{\pi}{J} = \frac{a_2}{l^2}T_{\perp\overline{\sigma}}{}^{\overline{\sigma}}, \tag{39}$$

where $\overset{\rightharpoonup}{T}_{\overline{\mu}} \equiv T_{\overline{\mu\nu}}{}^{\overline{\nu}}$, and a tensor with two indices contained in the bracket $\langle\rangle$ denotes that the tensor is symmetric-traceless with respect to the two indices. If the parameters take on any of the critical values: $2a_1 + a_2 = 0$, $a_1 + 2a_3 = 0$, $a_1 = 0$, or $a_2 = 0$, the Lagrangian becomes singular with respect to some velocities $\dot{e}_a{}^\mu$ (or equivalently, with respect to $T_{\perp\overline{\nu}}{}^\mu$), thus one obtains the following PIC: $\phi_{\overline{\mu}\perp} \approx 0$, $\overset{\wedge}{\phi}_{\overline{\mu\nu}} \approx 0$, $\overset{\sim}{\phi}_{\overline{\mu\nu}} \approx 0$, and/or $\phi \approx 0$, respectively.

The curvature momenta $\pi_{\overline{\sigma}\mu\nu}$ can be decomposed into six irreducible parts:

$$\pi_{\overline{\sigma}\mu\nu} = \pi_{\overline{\sigma\mu\nu}} + 2\pi_{\overline{\sigma}\perp[\overline{\mu}}n_{\nu]}, \tag{40}$$

and

$$\pi_{\overline{\mu\nu}\perp} = \overset{\wedge}{\pi}_{\overline{\mu\nu}\perp} + \overset{\sim}{\pi}_{\overline{\mu\nu}\perp} + \frac{1}{3}\eta_{\overline{\mu\nu}}\pi_\perp, \tag{41}$$

$$\pi_{\overline{\sigma\mu\nu}} = -\frac{1}{6}\epsilon_{\sigma\mu\nu\perp}{}^{\mathrm{P}}\pi + \overset{\rightharpoonup}{\pi}_{[\overline{\mu}}\eta_{\overline{\nu}]\overline{\sigma}} + \frac{4}{3}{}^{\mathrm{T}}\pi_{\overline{\sigma}[\overline{\mu\nu}]}. \tag{42}$$

Identifying the irreducible parts of the curvature momenta (14), one finds

$${}^{\mathrm{P}}\phi \equiv \frac{{}^{\mathrm{P}}\pi}{J} - \frac{1}{\kappa}(b_2 - b_3){}^{\mathrm{P}}R_{\circ\perp} = \frac{1}{\kappa}(b_2 + b_3){}^{\mathrm{P}}R_{\perp\circ}, \tag{43}$$

$$\overset{\rightharpoonup}{\phi}_{\overline{\mu}} \equiv \frac{\overset{\rightharpoonup}{\pi}_{\overline{\mu}}}{J} + \frac{1}{\kappa}(b_4 - b_5)R_{\overline{\mu}\perp} = -\frac{1}{\kappa}(b_4 + b_5)R_{\perp\overline{\mu}}, \tag{44}$$

$${}^{\mathrm{T}}\phi_{\overline{\sigma\mu\nu}} \equiv \frac{{}^{\mathrm{T}}\pi_{\overline{\sigma\mu\nu}}}{J} + \frac{1}{\kappa}(b_1 - b_2){}^{\mathrm{T}}R_{\overline{\mu\nu\sigma}\perp} = \frac{1}{\kappa}(b_1 + b_2){}^{\mathrm{T}}R_{\perp\overline{\sigma\mu\nu}}, \tag{45}$$

$$\phi_\perp \equiv \frac{\pi_\perp}{J} - \frac{3a_0}{l^2} - \frac{1}{2\kappa}(b_4 - b_6)\underline{R} = \frac{1}{\kappa}(b_4 + b_6)R_{\perp\perp}, \tag{46}$$

$$\overset{\wedge}{\phi}_{\overline{\mu\nu}\perp} \equiv \frac{\overset{\wedge}{\pi}_{\overline{\mu\nu}\perp}}{J} - \frac{1}{\kappa}(b_2 - b_5)\underline{R}_{[\overline{\mu\nu}]} = \frac{1}{\kappa}(b_2 + b_5)R_{\perp[\overline{\mu\nu}]\perp}, \tag{47}$$

$$\overset{\sim}{\phi}_{\overline{\mu\nu}\perp} \equiv \frac{\overset{\sim}{\pi}_{\overline{\mu\nu}\perp}}{J} - \frac{1}{\kappa}(b_1 - b_4)\underline{R}_{\langle\overline{\mu\nu}\rangle} = \frac{1}{\kappa}(b_1 + b_4)R_{\perp\langle\overline{\mu\nu}\rangle\perp}, \tag{48}$$

where ${}^{\mathrm{P}}R_{\circ\perp} := \epsilon^{\mu\nu\sigma\perp}R_{\overline{\mu\nu\sigma}\perp}$, ${}^{\mathrm{P}}R_{\perp\circ} := \epsilon^{\mu\nu\sigma\perp}R_{\perp\overline{\mu\nu\sigma}}$, $\underline{R}_{\overline{\mu\nu}} := R_{\overline{\sigma\mu\nu}}{}^{\overline{\sigma}}$, and $\underline{R} := \underline{R}_{\overline{\mu}}{}^{\overline{\mu}}$. By a similar argument as used above, for various degenerate parameter combinations one can obtain any of the six expressions of (43-48) as PIC's. The relations between the critical parameter combinations and the constraints are summarized in Table 1.

Blagojević and Nikolić found that, for the generic PGT, there are only primary and secondary if-constraints (no tertiary constraints). The Poisson bracket (PB) between either (i) appropriately paired primaries, or (ii) a primary and the secondary generated by its preservation, was found to be generally non-vanishing, hence they form a second-class pair. More specifically, the value of these PB's are just the (generally non-vanishing) constant "mass" parameter combinations of Table 1 plus some field-dependent terms of nonlinear origin. In Sec. 5 we will discuss the effects on the constraint classification due to these nonlinear terms.

Table 1. Primary if-constraints, critical parameter values and masses

J^p	Kinetic Parameter Combinations		Constraints		Mass Parameter Combinations
0^+	(a)	a_2	$\phi,$	χ	$a_0,\, 2a_0+a_2$
	(b)	b_4+b_6	$\phi_\perp,$	$\chi_\perp$	
1^+	(a)	a_1+2a_3	$\hat{\phi}_{\overline{\mu\nu}},$	$\hat{\chi}_{\overline{\mu\nu}}$	$a_1-a_0,\, \dfrac{a_0}{2}+a_3$
	(b)	b_2+b_5	$\hat{\phi}_{\overline{\mu\nu}\perp},$	$\hat{\chi}_{\overline{\mu\nu}\perp}$	
2^+	(a)	a_1	$\tilde{\phi}_{\overline{\mu\nu}},$	$\tilde{\chi}_{\overline{\mu\nu}}$	$a_0,\, a_1-a_0$
	(b)	b_1+b_4	$\tilde{\phi}_{\overline{\mu\nu}\perp},$	$\tilde{\chi}_{\overline{\mu\nu}\perp}$	
1^-	(a)	$2a_1+a_2$	$\phi_{\overline{\mu}\perp},$	$\chi_{\overline{\mu}\perp}$	$a_1-a_0,\, 2a_0+a_2$
	(b)	b_4+b_5	$\vec{\phi}_{\overline{\mu}},$	$\vec{\chi}_{\overline{\mu}}$	
0^-		b_2+b_3	${}^{\mathrm{P}}\phi,$	${}^{\mathrm{P}}\chi$	$\dfrac{a_0}{2}+a_3$
2^-		b_1+b_2	${}^{\mathrm{T}}\phi_{\overline{\sigma\mu\nu}},$	${}^{\mathrm{T}}\chi_{\overline{\sigma\mu\nu}}$	a_1-a_2

In order to treat all such possibilities in a concise way, the singular function

$$\frac{\lambda(x)}{x} \equiv \begin{cases} \dfrac{1}{x}, & x \neq 0, \\ 0, & x = 0, \end{cases} \tag{49}$$

was introduced. The PIC's in the total Hamiltonian density (33) can be given in the form

$$u^A \phi_A = (u \cdot \phi)^T + (u \cdot \phi)^R, \tag{50}$$

where

$$\begin{aligned} (u \cdot \phi)^T \;\; \equiv \;\; & [1-\lambda(2a_1+a_2)]u^{\overline{\mu}\perp}\phi_{\overline{\mu}\perp} + [1-\lambda(a_1)]\tilde{u}^{\overline{\mu\nu}}\tilde{\phi}_{\overline{\mu\nu}} \\ & +[1-\lambda(a_1+2a_3)]\hat{u}^{\overline{\mu\nu}}\hat{\phi}_{\overline{\mu\nu}} + \frac{1}{3}[1-\lambda(a_2)]u\phi \end{aligned} \tag{51}$$

and

$$\begin{aligned} (u \cdot \phi)^R \;\; \equiv \;\; & \frac{1}{6}[1-\lambda(b_2+b_3)]{}^{\mathrm{P}}u{}^{\mathrm{P}}\phi + \frac{4}{3}[1-\lambda(b_1+b_2)]{}^{\mathrm{T}}u^{\overline{\sigma\mu\nu}}\,{}^{\mathrm{T}}\phi_{\overline{\sigma\mu\nu}} \\ & +[1-\lambda(b_4+b_5)]\vec{u}^{\overline{\mu}}\vec{\phi}_{\overline{\mu}} + 2[1-\lambda(b_2+b_5)]\hat{u}^{\overline{\mu\nu}\perp}\hat{\phi}_{\overline{\mu\nu}\perp} \\ & +2[1-\lambda(b_1+b_4)]\tilde{u}^{\overline{\mu\nu}\perp}\tilde{\phi}_{\overline{\mu\nu}\perp} + \frac{2}{3}[1-\lambda(b_4+b_6)]u^{\perp}\phi_{\perp}. \end{aligned} \tag{52}$$

The super-Hamiltonian $\mathcal{H}_\perp$ in $\mathcal{H}_{\rm can}$ (28) then turns out to be of the form

$$\mathcal{H}_\perp = \mathcal{H}_\perp^T + \mathcal{H}_\perp^R, \tag{53}$$

with

$$\mathcal{H}^T_\perp = J\frac{l^2}{2}\left[\frac{3\lambda(2a_1+a_2)}{2a_1+a_2}\phi_{\overline{\mu}\perp}\phi^{\overline{\mu}\perp} + \frac{3\lambda(a_1+2a_3)}{a_1+2a_3}\hat{\phi}_{\overline{\mu\nu}}\hat{\phi}^{\overline{\mu\nu}} + \frac{\lambda(a_1)}{a_1}\tilde{\phi}_{\overline{\mu\nu}}\tilde{\phi}^{\overline{\mu\nu}} + \frac{\lambda(a_2)}{3a_2}\phi^2\right] + J\underline{\mathcal{L}}^T - n^\mu\nabla_a\pi^a{}_\mu, \tag{54}$$

$$\mathcal{H}^R_\perp = J\kappa\left[\frac{\lambda(b_2+b_3)}{24(b_2+b_3)}{}^{\mathrm{P}}\phi^2 + \frac{\lambda(b_4+b_5)}{4(b_4+b_5)}\vec{\phi}_{\overline{\mu}}\vec{\phi}^{\overline{\mu}} + \frac{\lambda(b_1+b_2)}{3(b_1+b_2)}{}^{\mathrm{T}}\!\phi_{\overline{\sigma\mu\nu}}{}^{\mathrm{T}}\!\phi^{\overline{\sigma\mu\nu}} + \frac{\lambda(b_2+b_5)}{2(b_2+b_5)}\hat{\phi}_{\overline{\mu\nu}\perp}\hat{\phi}^{\overline{\mu\nu}\perp} + \frac{\lambda(b_1+b_4)}{2(b_1+b_4)}\tilde{\phi}_{\overline{\mu\nu}\perp}\tilde{\phi}^{\overline{\mu\nu}\perp} + \frac{\lambda(b_4+b_6)}{6(b_4+b_6)}\phi_\perp\phi^\perp\right] + J\underline{\mathcal{L}}^R, \tag{55}$$

where

$$\underline{\mathcal{L}}^T = \frac{1}{12l^2}\left[(2a_1+a_3)T_{\overline{\nu\sigma}\mu}T^{\overline{\nu\sigma}\mu} + 2(a_1-a_3)T_{\overline{\nu\sigma\mu}}T^{\overline{\mu\sigma\nu}} - 2(a_1-a_2)\vec{T}_{\overline{\mu}}\vec{T}^{\overline{\mu}}\right], \tag{56}$$

$$\underline{\mathcal{L}}^R = -c_0R_{\overline{\tau\sigma}\mu\nu}R^{\overline{\tau\sigma}\mu\nu} - c_1R_{\overline{\tau\sigma\mu}\nu}R^{\overline{\tau\mu\sigma}\nu} - c_2R_{\overline{\tau\sigma\mu\nu}}R^{\overline{\mu\nu\tau\sigma}} - c_3(\underline{R}_{\overline{\mu\nu}}\underline{R}^{\overline{\mu\nu}} + R_{\overline{\mu}\perp}R^{\overline{\mu}\perp}) - c_4\underline{R}_{\overline{\mu\nu}}\underline{R}^{\overline{\nu\mu}} - c_5\underline{R}^2 - \frac{a_0}{2l^2}\underline{R} + \Lambda, \tag{57}$$

here Λ is the cosmological constant. The relations between the constants c_i's and the b_i's are given by

$$\begin{aligned} c_0 &= -\frac{1}{24\kappa}(2b_1+3b_2+b_3),\\ c_1 &= -\frac{1}{6\kappa}(b_1-b_3),\\ c_2 &= -\frac{1}{24\kappa}(2b_1-3b_2+b_3),\\ c_3 &= \frac{1}{4\kappa}(b_1+b_2-b_4-b_5),\\ c_4 &= \frac{1}{4\kappa}(b_1-b_2-b_4+b_5),\\ c_5 &= -\frac{1}{24\kappa}(2b_1-3b_4+b_6). \end{aligned} \tag{58}$$

We now can use the formulations above to construct the total Hamiltonian under different critical parameter values. For example, if $2a_1+a_2$ vanishes, this results in the relation in Eq. (36) becomes the constraint $\phi_{\overline{\mu}\perp} \equiv \frac{\pi_{\overline{\mu}\perp}}{J} + \frac{a_1}{l^2}\vec{T}_{\overline{\mu}} \approx 0$. Therefore $[1-\lambda(2a_1+a_2)]u^{\overline{\mu}\perp}\phi_{\overline{\mu}\perp} = u^{\overline{\mu}\perp}\phi_{\overline{\mu}\perp}$ in the torsion if-constraint part (51); at the same time, $\frac{3\lambda(2a_1+a_2)}{2a_1+a_2}\phi_{\overline{\mu}\perp}\phi^{\overline{\mu}\perp} = 0$ in $\mathcal{H}^T_\perp$ (54), by referring to the singular function (49). On the other hand, if $2a_1+a_2 \neq 0$, the relation (36) stands and $\frac{3\lambda(2a_1+a_2)}{2a_1+a_2}\phi_{\overline{\mu}\perp}\phi^{\overline{\mu}\perp} = \frac{3}{2a_1+a_2}\phi_{\overline{\mu}\perp}\phi^{\overline{\mu}\perp}$ and $[1-\lambda(2a_1+a_2)]u^{\overline{\mu}\perp}\phi_{\overline{\mu}\perp} = 0$. This process can be performed term by term with different critical parameter combinations. It is very beneficial for the analysis of the PGT with general or specific values of the parameters.

4 The Linearized Theories of PGT

It is difficult to tell whether the PGT is viable under physically reasonable circumstances because of its complications (Indeed deep analysis has still left the issue obscure). Accordingly, the weak field approximation approach was the first and preliminary method applied to the PGT to study its properties. Linearization gives us insight into the viability of the propagating modes in the PGT. In the linearized theory we usually impose the physical requirements that the propagating modes carry positive energy and do not propagate faster than light. Theories with higher spin interactions are notorious for having such problems[18]. It was noted that interactions involving spins ≥ 1 were especially vulnerable to having their characteristics outside the metric light cone, thereby permitting acausal modes of signal propagation[18]. But linearization gets rid of nonlinear terms. One can tentatively forget the complex and focus on the structure of the theory. It is efficient for quickly eliminating the obviously unfit theories.

In the linearized PGT, in addition to the graviton there are possible propagating modes with $\text{spin}^{\text{parity}} = J^p = 2^+, 2^-, 1^+, 1^-, 0^+, 0^-$. The requirements for the possible propagating "massive" modes are essentially $v < c$ (no tachyons) and $E \geq 0$ (no ghosts). Sezgin and van Nieuwenhuizen[5,36] studied the linearized PGT in the early 80's. They applied the so-called "spin-projection operator formalism"[6] to the PGT and discovered that at least three of the massive modes have to be eliminated. By their analysis there are twelve such solutions. Their results are listed in Table 2. Hayashi and Shirafuji used the weak field approximation and the Klein-Gordon equation to investigate the particle spectrum of PGT[7]. They gave tables of all the maximal dynamic torsion modes. Their analysis confirmed that these requirements preclude the possibility of more than 3 dynamic propagating torsion modes. Consequently, there are many distinct cases with the torsion modes showing different behavior: some modes frozen and some dynamic.

Table 2. Parameter choices for ghost- and tachyon-free gravity Lagrangian determined by Sezgin and van Nieuwenhuizen.

	Parameter choices	Particle content
(1)	$a_1 + 2a_3 = b_1 + b_2 = b_4 + b_5 = 0$	$2^+, 0^+, 0^-$
(2)	$a_1 + 2a_3 = b_1 + b_2 = b_1 + b_4 = 0$	$1^-, 0^+, 0^-$
(3)	$a_1 + 2a_3 = b_1 + b_2 = b_4 + b_6 = 0$	$2^+, 1^-, 0^-$
(4)	$2a_1 + a_2 = b_1 + b_2 = b_1 + b_4 = 0$	$1^+, 0^+, 0^-$
(5)	$2a_1 + a_2 = b_1 + b_2 = b_4 + b_6 = 0$	$2^+, 1^+, 0^-$
(6)	$a_1 = b_1 + b_2 = b_4 + b_5 = 0$	$1^+, 0^+, 0^-$
(7)	$a_1 = b_1 + b_2 = b_4 + b_6 = 0$	$1^+, 1^-, 0^-$
(8)	$a_1 + 2a_3 = b_2 + b_3 = b_1 + b_4 = 0$	$2^-, 1^-, 0^+$
(9)	$2a_1 + a_2 = b_2 + b_3 = b_1 + b_4 = 0$	$2^-, 1^+, 0^+$
(10)	$2a_1 + a_2 = b_2 + b_5 = b_1 + b_4 = 0$	$2^-, 0^+, 0^-$
(11)	$a_1 = a_2 = b_1 + b_2 = 0$	$1^+, 0^-$
(12)	$b_1 + b_2 = b_4 + b_5 = b_2 + b_5 = 0$	$0^+, 0^-$

The linearized method was also applied to the cases with "massless" modes[36,37,38]. The propagating massless modes can then have gauge freedom. Kuhfuss and Nitsch[1] obtained some

very restrictive PGT parameter restrictions by using the spin-projection operator formalism and a "no fourth order pole requirement". Their restrictions are

$$
\begin{aligned}
&\Lambda = 2a_1 + a_2 = a_1 + 2a_2 = 0, \\
&b_1 + b_4 = b_4 + b_6 = b_2 + b_5 = 0, \\
&a_1 = a_0 > 0, \quad b_2, b_3 \in \mathrm{R} \text{ (arbitrary)}.
\end{aligned}
\tag{59}
$$

This parameter choice basically activates the spin-0^- and spin-2^- modes (refer to Table 1) and suppresses all the other modes.* These two propagating modes are both massless.

The idea of "second-class pairs" of constraints introduced in Sec. 3 is useful in analyzing the linearized PGT. It shows that the Hamiltonian analysis is in complete accord with the linearized theory analysis of the propagating modes[5,7,36]. The cases where one or more of the "masses" vanish have also been analyzed by Blagojević's group using the Hamiltonian procedure—although only to linear order[39]. The results are in accord with the linearized theory analysis of propagating massless modes and their associated extra gauge symmetries[1,36,37,38].

It is satisfying that all of the aforementioned results are essentially consistent with one another, however, we should beware of the limitations of linearization. Although we may obtain a relatively quick understanding of the system via the linearization of the theory, the result from the full theory, especially a highly nonlinear system, does not necessarily keep the same traits as its linearization. This fact has made us suspicious regarding these pioneers' proclamations. Especially, we view the conclusions of linearized theory for the massless modes as rather likely requiring qualitative revision from nonlinear effects and hence we regard them as much less firm than those for the massive modes. Consequently, we don't know how much confidence to place in their conclusions. Our reservations concerning the linearized propagating massless modes in no way affect the main conclusions from the survey of the linearized PGT. However, it seems likely that including the nonlinear effect terms would modify the Poisson brackets of the constraints, converting many of the vanishing brackets to non-vanishing ones. In this way a constraint which was first-class, with vanishing (on shell) brackets with all other constraints, would move to the second-class indicating that a gauge symmetry of the linearized theory was broken in the nonlinear theory. Hence we expect that the present understanding of "massless" PGT modes and their extra gauge symmetries, being based on a linearized approximation which appears inadequate, will require substantial revisions.

Aside from nonlinear effects, the analyses of the linearized theories of PGT offers the identification of the torsion modes, the critical kinetic parameter combinations, the mass terms and other useful relations. The critical kinetic and mass parameter combinations are given in Table 1. The necessary signature conditions for no ghosts and no tachyons are presented in Table 3.

5 Hamiltonian Analysis of Propagating Torsion Modes

The works discussed in the last section showed that there are still many viable linearized theories of PGT. In the other words, the linearized theories with these parameter choices have good behavior in the weak field approximation. The next step is to consider if the full nonlinear theories with the same parameter choices still survive. It is quite important that the whole structure of the theory stays the same before and after linearization. (Theories that do not

*Our understanding of the propagating modes from this parameter choice is different from Ref. 1, further study is needed to clarify this point.

Table 3. No ghosts and no tachyons conditions for the torsion modes

J^p	Energy>0 and Mass$^2>0$
0^+	$b_4+b_6>0, \quad a_0a_2(a_2+2a_0)<0$
0^-	$b_2+b_3<0, \quad a_3+a_0/2<0$
1^+	$b_2+b_5>0, \quad (a_1+2a_3)(a_1-a_0)(a_3+a_0/2)>0$
1^-	$b_4+b_5<0, \quad (2a_1+a_2)(a_1-a_0)(a_2+2a_0)<0$
2^+	$a_0<0, \quad b_1+b_4>0, \quad a_0a_1(a_1-a_0)>0$
2^-	$b_1+b_2<0, \quad a_1-a_0<0$

have this character are inherently nonlinear, and are much more difficult to analyze. We are very reluctant to seriously entertain such theories unless we have no alternative.) In particular we should have no change in the number and type of constraints, for that is what determines the number of gauge freedoms and the number of dynamic physical degrees of freedom. If the answer is affirmative, the theory can be smoothly transformed from nonlinearity to linearity and vice versa with its properties unchanged. If the answer is negative, the properties of the theory would change fundamentally upon linearization. In fact it has been argued that such changes in the properties are accompanied by the occurrence of serious physical problems, in particular anomalous characteristics[40,41]. Hence a sound case of the PGT with "good" parameter choices should not exhibit such a change of properties behavior when passing to the weak field approximation. Thus this becomes a criterion that determines if some specific parameter choice is good enough, or whether it should be regarded as non-viable.

We are going to investigate the viability of the individual simple propagating spin modes and two better-behaved cases. Here "simple" indicates that all parameters will be set to vanish except the key ones. Such severe conditions could be generalized. For example, in order to suppress all except the 0^+ and 0^- modes, according Table 1. our condition could be relaxed to require only $b_1=-b_2=-b_4=b_5$ instead of our severe "simple" choice $b_1=b_2=b_4=b_5=0$. There are two reasons not to adopt the "relaxed" versions: (a) it would much complicate the analysis, (b) the relaxed parameter choices do not essentially enhance the viability of the theory; instead, they could even ruin a case which is well-behaved under the simple version. We can readily see this from the primary Poisson matrices of all the cases.

It is not the purpose of this paper to examine viability in the full nonlinear domain of all the "viable" linearized theories of PGT. It seems appropriate to first try to illustrate the influence of nonlinearity in these theories and to investigate the effects in order to get some understanding. Nevertheless, we believe that the phenomena resulting from nonlinearity in the cases discussed in this paper are representative for all the other cases. We would like to emphasize the effects of nonlinearity on the change of the structure and gauges of the theory, so we will focus on the change of the type and/or number of the constraints and the consistency of the whole system caused by nonlinearity. These observations will be solid enough for an understanding of the differences before and after linearization.

In our investigations we found that the "constraint bifurcation" phenomena is a key evidence which determines if there is an anomalous characteristic. Whether the Poisson matrix has

constant rank or not, independent of the field values, is the criteria for knowing whether the phenomenon of constraint bifurcation occurs. A field dependent rank can happen only if the constraints are nonlinear. Strange behavior may occur as one approaches a point in the phase space where the rank of the Poisson matrix changes. At such points the number and type of constraints (and thus the number of gauges and physical degrees of freedom) are not constant. This phenomenon signals the presence of an acausal propagating mode in the system and is unacceptable theoretically[41]. Whether the Poisson matrix is singular can be judged by the positivity of its determinant. Since the matrix is usually big the concise form of its determinant is hard to obtain, so we appeal to the aid of the symbolic computing software `REDUCE`[42] for the calculations. From the result associated with all the constraints we can decide the positivity of the determinant and thus the possibility of constraint bifurcation.

5.1 Simple spin-one and spin-two cases

The method of the analysis is simple conceptually: list all of the constraints and compare the differences of the structure before and after linearization. The Poisson matrix is formed containing the non-zero Poisson Brackets (PB), then checked for change under linearization. Due to the complications, the secondary constraints (SC) will not be expressed explicitly. Instead we will reason out the differences based on the result of the SC's linearization in certain situations.

5.1.1 The simple spin-1^+ case

$\overset{\wedge}{\pi}_{\overline{\mu\nu}\perp}$ corresponds to the spin-1^+ mode. The linearized theory indicates that the theory with only a spin-1^+ mode can be obtained with the simple parameter choice:

$$\begin{aligned} &a_0 \neq 0, \quad a_3 \neq 0, \quad b_5 \neq 0, \\ &a_1 = a_2 = 0, \\ &b_1 = b_2 = b_3 = b_4 = b_6 = 0. \end{aligned} \tag{60}$$

With such a parameter choice there are twenty-one PIC's. The PIC's are as follows:

$$\begin{array}{lllllllll} \phi &\equiv& \dfrac{\pi}{J} \approx 0, & \overset{\sim}{\phi}_{\overline{\mu\nu}} &\equiv& \dfrac{\overset{\sim}{\pi}_{\overline{\mu\nu}}}{J} \approx 0, & \phi_{\overline{\mu}\perp} &\equiv& \dfrac{\pi_{\overline{\mu}\perp}}{J} \approx 0, \\ {}^{\mathrm{P}}\phi &\equiv& \dfrac{{}^{\mathrm{P}}\pi}{J} \approx 0, & {}^{\mathrm{T}}\phi_{\overline{\sigma\mu\nu}} &\equiv& \dfrac{{}^{\mathrm{T}}\pi_{\overline{\sigma\mu\nu}}}{J} \approx 0, & & & \\ \phi_{\perp} &\equiv& \dfrac{\pi}{J} - \dfrac{3a_0}{l^2} \approx 0, & \overset{\sim}{\phi}_{\overline{\mu\nu}\perp} &\equiv& \dfrac{\overset{\sim}{\pi}_{\overline{\mu\nu}\perp}}{J} \approx 0. & & & \end{array} \tag{61}$$

The non-zero PB's are

$$\{\phi_{\overline{\mu}\perp}, \phi'_{\overline{\mu}\perp}\} = 2\frac{\delta_{xx'}}{J^2}\overset{\wedge}{\pi}_{\overline{\mu\nu}}, \tag{62}$$

$$\{\overset{\sim}{\phi}_{\overline{\mu\nu}}, \overset{\sim}{\phi}'_{\overline{\tau\sigma}}\} = 2\frac{\delta_{xx'}}{J^2}\overset{\wedge}{\pi}_{(\overline{\tau}(\overline{\mu}\eta_{\overline{\nu}})\overline{\sigma})}, \tag{63}$$

$$\{{}^{\mathrm{P}}\phi, \phi'_{\overline{\mu}\perp}\} = 2\frac{\delta_{xx'}}{J^2}\epsilon_{\mu}{}^{\nu\sigma\perp}\overset{\wedge}{\pi}_{\overline{\nu\sigma}\perp}, \tag{64}$$

$$\{\phi_{\perp}, \phi'\} = -\frac{\delta_{xx'}}{J}\frac{6a_0}{l^2}, \tag{65}$$

$$
\begin{aligned}
\{\phi_\perp, \phi'_{\overline{\mu}\perp}\} &= -\frac{\delta_{xx'}}{J^2}\overset{\rightharpoonup}{\pi}_{\overline{\mu}}, && (66)\\
\{{}^{T}\!\phi_{\overline{\sigma\mu\nu}}, \phi'_{\overline{\tau}\perp}\} &= \frac{\delta_{xx'}}{J^2}{}^{T}\!\left(\eta_{\overline{\tau\nu}}\overset{\wedge}{\pi}_{\overline{\sigma\mu}\perp}\right) && (67)\\
\{{}^{T}\!\phi_{\overline{\sigma\mu\nu}}, \tilde{\phi}'_{\overline{\rho\tau}}\} &= \frac{1}{2}\frac{\delta_{xx'}}{J^2}\left[\overset{\rightharpoonup}{\pi}_{\overline{\mu}}\eta_{\overline{\nu}\langle\overline{\rho}}\eta_{\overline{\tau}\rangle\overline{\sigma}} - {}^{T}\!\left(\overset{\rightharpoonup}{\pi}_{\overline{\sigma}}\eta_{\overline{\mu}\langle\overline{\rho}}\eta_{\overline{\tau}\rangle\overline{\nu}}\right)\right], && (68)\\
\{\tilde{\phi}_{\overline{\mu\nu}\perp}, \phi'_{\overline{\sigma}\perp}\} &= \frac{1}{2}\frac{\delta_{xx'}}{J^2}\overset{\rightharpoonup}{\pi}_{\langle\overline{\mu}}\eta_{\overline{\nu}\rangle\overline{\sigma}}, && (69)\\
\{\tilde{\phi}_{\overline{\mu\nu}\perp}, \tilde{\phi}'_{\overline{\tau\sigma}}\} &= \frac{\delta_{xx'}}{J}\left[\frac{1}{J}\eta_{(\overline{\mu}(\overline{\tau}}\overset{\wedge}{\pi}_{\overline{\sigma})\overline{\nu})} + \frac{a_0}{l^2}\eta_{\overline{\tau}\langle\overline{\mu}}\eta_{\overline{\nu}\rangle\overline{\sigma}}\right]. && (70)
\end{aligned}
$$

With these PB's we can construct the primary Poisson matrix with the following form:

$$
\begin{array}{ccc|ccccccc|}
 & & & \overbrace{}^{1} & \overbrace{}^{3} & \overbrace{}^{5} & \overbrace{}^{1} & \overbrace{}^{1} & \overbrace{}^{5} & \overbrace{}^{5} \\
 & & & \phi & \phi_{\overline{\mu}\perp} & \tilde{\phi}_{\overline{\mu\nu}} & \phi_\perp & {}^{P}\!\phi & \tilde{\phi}_{\overline{\mu\nu}\perp} & {}^{T}\!\phi_{\overline{\sigma\mu\nu}} \\
\hline
1 & \{ & \phi & \bigcirc & \bigcirc & \bigcirc & \eta & \bigcirc & \bigcirc & \bigcirc \\
3 & \{ & \phi_{\overline{\mu}\perp} & \bigcirc & \pi & \bigcirc & \pi & \pi & \pi & \pi \\
5 & \{ & \tilde{\phi}_{\overline{\mu\nu}} & \bigcirc & \bigcirc & \pi & \bigcirc & \bigcirc & \eta+\pi & \pi \\
1 & \{ & \phi_\perp & \eta & \pi & \bigcirc & & & & \\
1 & \{ & {}^{P}\!\phi & \bigcirc & \pi & \bigcirc & & & \bigcirc & \\
5 & \{ & \tilde{\phi}_{\overline{\mu\nu}\perp} & \bigcirc & \pi & \eta+\pi & & & & \\
5 & \{ & {}^{T}\!\phi_{\overline{\sigma\mu\nu}} & \bigcirc & \pi & \pi & & & &
\end{array}
\qquad (71)
$$

where η's represent the constant terms and π's represent the terms involving the variables. The details of these terms can be found in Eq. (62)-(70). The super-Hamiltonian density is deduced by applying the parameter choice (60) to (53). It is

$$
\mathcal{H}_\perp^{1^+} = \mathcal{H}_\perp^{R1^+} + \mathcal{H}_\perp^{T1^+}, \qquad (72)
$$

where

$$
\begin{aligned}
\mathcal{H}_\perp^{R1^+} &= \frac{1}{2}\overset{\rightharpoonup}{\pi}_{\overline{\mu}}\left[\frac{\kappa}{2b_5}\frac{\overset{\rightharpoonup}{\pi}^{\overline{\mu}}}{J} + R^{\overline{\mu}\perp}\right] - \overset{\wedge}{\pi}_{\overline{\mu\nu}\perp}\left[\frac{\kappa}{2b_5}\frac{\overset{\wedge}{\pi}^{\overline{\mu\nu}}{}_{\perp}}{J} + \underline{R}^{[\overline{\mu\nu}]}\right] - \frac{a_0}{l^2}J\underline{R}, && (73)\\
\mathcal{H}_\perp^{T1^+} &= \frac{1}{2}\overset{\wedge}{\pi}_{\overline{\mu\nu}}\left[\frac{3l^2}{2a_3}\frac{\overset{\wedge}{\pi}^{\overline{\mu\nu}}}{J} + T^{\overline{\mu\nu}\perp}\right] + \frac{a_3}{24l^2}J{}^{P}T^2 - n^\mu\nabla_a\pi^a{}_\mu. && (74)
\end{aligned}
$$

Within the phase surface of primary constraints, the super-momenta (30) and the Lorentz rotation parts (31) lead to the following constraints:

$$
\begin{aligned}
\mathcal{H}_{\overline{\mu}}^{1^+} &\longrightarrow \eta_{\overline{\mu\nu}}\nabla_{\overline{\sigma}}\frac{\overset{\wedge}{\pi}^{\overline{\nu\sigma}}}{J} + R_{\overline{\mu\nu\sigma}\perp}\frac{\overset{\wedge}{\pi}^{\overline{\nu\sigma}\perp}}{J} - \frac{1}{2J}\underline{R}_{\overline{\mu\nu}}\overset{\rightharpoonup}{\pi}^{\overline{\nu}} - \frac{a_0}{l^2}R_{\overline{\mu}\perp}\\
&\quad + \frac{3}{2J}\overset{\wedge}{\pi}_{\overline{\mu\nu}}\overset{\rightharpoonup}{T}^{\overline{\nu}} - \frac{2}{3J}{}^{T}T_{\overline{\nu\sigma\mu}}\overset{\wedge}{\pi}^{\overline{\nu\sigma}} - \frac{1}{6J}\epsilon_{\mu\nu\sigma\perp}{}^{P}T\overset{\wedge}{\pi}^{\overline{\nu\sigma}} \approx 0, && (75)\\
\mathcal{H}_{\overline{\mu}\perp}^{1^+} &\longrightarrow \frac{\overset{\wedge}{\pi}_{\overline{\mu\nu}\perp}}{J}\overset{\rightharpoonup}{T}^{\overline{\nu}} - \frac{a_0}{l^2}\overset{\rightharpoonup}{T}_{\overline{\mu}} + \frac{1}{J}\overset{\rightharpoonup}{\pi}_{[\overline{\mu}}\nabla_{\overline{\nu}]}n^\nu - \delta_{\overline{\mu}}{}^{\sigma}\nabla_{\overline{\nu}}\frac{\overset{\wedge}{\pi}^{\overline{\nu}}{}_{\overline{\sigma}\perp}}{J} \approx 0, && (76)
\end{aligned}
$$

$$\mathcal{H}^{1^+}_{\overline{\mu\nu}} \;\longrightarrow\; \frac{\hat{\pi}_{\overline{\mu\nu}}}{J} + \frac{a_0}{2l^2}T_{\overline{\mu\nu}\perp} + \frac{1}{2J}\vec{\pi}_{[\overline{\mu}}\vec{T}_{\overline{\nu}]} + \frac{1}{2}\delta^{\sigma}{}_{[\overline{\mu}}\nabla_{\overline{\nu}]}\frac{\vec{\pi}_{\overline{\sigma}}}{J} - \frac{1}{J}\hat{\pi}^{\overline{\sigma}}{}_{[\overline{\mu}|\perp}\nabla_{\overline{\sigma}}n_{|\nu]} \approx 0. \tag{77}$$

It is essential to compare the nonlinear spin-1^+ case with its linearized version. At first let us describe briefly the analysis of the linear spin-1^+ case. After linearization, the remaining non-zero PB's turn out to be Eqs. (65), (70) in which only the zero-order (constant) terms exist, i.e.,

$$\{\phi^{\flat}_{\perp}, \phi^{\flat\prime}\} \;\approx\; \delta_{xx'}\frac{6a_0}{l^2}, \tag{78}$$

$$\{\tilde{\phi}^{\flat}_{\overline{\mu\nu}\perp}, \tilde{\phi}^{\flat\prime}_{\overline{\tau\sigma}}\} \;\approx\; -\delta_{xx'}\frac{a_0}{l^2}\eta^{\flat}_{\overline{\tau}\langle\overline{\mu}}\eta^{\flat}_{\overline{\nu}\rangle\overline{\sigma}}, \tag{79}$$

where the superscript $\flat$ indicates linearization. With the linearized PB's we can construct the constant primary Poisson matrix:

$$\begin{array}{lll|ccccccc|}
 & & & \overbrace{\phi^{\flat}}^{1} & \overbrace{\phi^{\flat}_{\overline{\mu}\perp}}^{3} & \overbrace{\tilde{\phi}^{\flat}_{\overline{\mu\nu}}}^{5} & \overbrace{\phi^{\flat}_{\perp}}^{1} & \overbrace{{}^{P}\phi^{\flat}}^{1} & \overbrace{\tilde{\phi}^{\flat}_{\overline{\mu\nu}\perp}}^{5} & \overbrace{{}^{T}\phi^{\flat}_{\overline{\sigma\mu\nu}}}^{5} \\
1 & \{ & \phi^{\flat} & & & & \eta & \bigcirc & \bigcirc & \bigcirc \\
3 & \{ & \phi^{\flat}_{\overline{\mu}\perp} & & \bigcirc & & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
5 & \{ & \tilde{\phi}^{\flat}_{\overline{\mu\nu}} & & & & \bigcirc & \bigcirc & \eta & \bigcirc \\
1 & \{ & \phi^{\flat}_{\perp} & -\eta & \bigcirc & \bigcirc & & & & \\
1 & \{ & {}^{P}\phi^{\flat} & \bigcirc & \bigcirc & \bigcirc & & & \bigcirc & \\
5 & \{ & \tilde{\phi}^{\flat}_{\overline{\mu\nu}\perp} & \bigcirc & \bigcirc & -\eta & & & & \\
5 & \{ & {}^{T}\phi^{\flat}_{\overline{\sigma\mu\nu}} & \bigcirc & \bigcirc & \bigcirc & & & &
\end{array} \tag{80}$$

where η represents the constants in Eq. (78) and (79). The consistency conditions guarantee that the linearized Lagrange multipliers, i.e., $u^{\flat}$, $u^{\flat\perp}$, $\tilde{u}^{\flat\overline{\mu\nu}}$, and $\tilde{u}^{\flat\overline{\mu\nu}\perp}$, are obtained. The nine linearized SC, $\chi^{\flat}_{\overline{\mu}\perp}$, ${}^{P}\chi^{\flat}$ and ${}^{T}\chi^{\flat}_{\overline{\sigma\mu\nu}}$, also show up when the consistency conditions of the linearized constraints $\phi^{\flat}_{\overline{\mu}\perp}$, ${}^{P}\phi^{\flat}$ and ${}^{T}\phi^{\flat}_{\overline{\sigma\mu\nu}}$ are required. Furthermore, the linearized SC $\chi^{\flat}_{\overline{\mu}\perp}$, ${}^{P}\chi^{\flat}$ and ${}^{T}\chi^{\flat}_{\overline{\sigma\mu\nu}}$ will become the counterparts to the second-class pairs of $\phi^{\flat}_{\overline{\mu}\perp}$, ${}^{P}\phi^{\flat}$ and ${}^{T}\phi^{\flat}_{\overline{\sigma\mu\nu}}$ respectively according to Ref. 23. These second-class pairs determine the multipliers $u^{\flat\overline{\mu}\perp}$, ${}^{P}u^{\flat}$ and $\tilde{u}^{\flat\overline{\mu\nu}}$ and the algorithm of the linearized case is terminated. In the PGT, there are forty dynamic variables coming from the sixteen tetrad components and the twenty-four connection components. The number of total variables counts eighty because the same number of canonical momenta accompany the variables. Nonetheless, constraints eliminate many unphysical variables. It is known that the super-Hamiltonian $\mathcal{H}_{\perp}$, super-momenta $\mathcal{H}_a$, and the Lorentz generators $\mathcal{H}_{\mu\nu}$ are ten first-class constraints. There are also ten sure first-class primary constraints. The total twenty first-class constraints (because of their gauge nature) offset forty variables. The degrees of freedom of the linearized case can be counted as $\frac{1}{2}(80[\text{total}]-40[\text{gauge}]-21[\text{PIC}]-9[\text{SC}]) = 5 = 3[1^+]+2[\text{GR}]$, a massive spin-$1^+$ field in addition of the usual graviton. For the sake of "ghost-free" the positivity of the kinetic energy density demands $a_3 < 0$ and $b_5/\kappa > 0$.

Notwithstanding the good behavior of the linearized case, the nonlinearity of the full spin-1^+ theory ruins its whole legitimacy. Generically, the Lagrange multipliers u, $u^{\perp}$, $\tilde{u}^{\overline{\mu\nu}}$, $\tilde{u}^{\overline{\mu\nu}\perp}$

can not be determined alone because the nonlinear terms are engaged (see Eq. (71)). Here we assume tentatively that there is no inconsistency appearing in the system during the process. The consistency conditions of the connection PIC's, $\phi_{\perp}$, ${}^{\mathrm{P}}\phi$, $\overset{\sim}{\phi}_{\overline{\mu\nu}\perp}$ and ${}^{\mathrm{T}}\phi_{\overline{\sigma\mu\nu}}$, will determine the multipliers of the tetrad PIC's, u, $u^{\overline{\mu}\perp}$ and $\widetilde{u}^{\overline{\mu\nu}}$, and give three SC's, $\chi_{[3]}$. Principally, the consistency conditions of the tetrad PIC's and the SC's $\chi_{[3]}$ will determine the multipliers, $u^{\perp}$, ${}^{\mathrm{P}}u$, $\widetilde{u}^{\overline{\mu\nu}\perp}$ and ${}^{\mathrm{T}}u^{\overline{\sigma\mu\nu}}$ according to the experience of the linearized case and terminate the process. The degrees of freedom of the full nonlinear case count as $\frac{1}{2}(80-40-21[\mathrm{PIC}]-3[\mathrm{SC}])=8=2+3[1^{+}]+3[1^{-}]$. This shows that the massive spin-1^{-} field is excited as well as the massive spin-1^{+} field since the other modes are already suppressed. The result is unacceptable from the point of view of the positivity of the kinetic energy density in the Hamiltonian density (refer to Eq. (73)).

One can learn the major difference just from the primary Poisson matrix. The rank of the linearized primary Poisson matrix (80) in this case is twelve, a constant, whereas the rank of the full nonlinear matrix (71) varies. Its rank is eighteen generically but it can be less than eighteen depending on some specific numerical values of the variables which could degenerate the system. We verified this point by using the symbolic computation software `REDUCE` to calculate the determinant of the matrix (71). The result shows that the determinant is not positive definite, which indicates that whether the system is singular depends on the numerical values of the variables. This situation shows the phenomenon of constraint bifurcation. It could even happen as the system dynamically evolves. Consider all of the possible initial values that give generic rank to the Poisson matrix. We see no mechanism that would guarantee that every one of these cases could not evolve to field values where the rank changes. Because of these features we regard this parameter choice as physically unacceptable.

5.1.2 The simple spin-1^{-} case

We next examine the simple spin-1^{-} case and find that it has similar problems as in the simple spin-1^{+} case. $\overset{\rightharpoonup}{\pi}_{\overline{\mu}}$ corresponds to the spin-1^{-} mode. The linearized theory indicates that the theory with only the spin-1^{-} mode is obtained if only a_0, a_2 and b_5 do not vanish. Under this parameter choice $\overset{\wedge}{\phi}_{\overline{\mu\nu}}$ instead of ϕ and $\phi_{\overline{\mu}\perp}$ become PIC's and the other PIC's remain just as for the spin-1^{+} case. The non-zero PB's are

$$\{{}^{\mathrm{P}}\phi, \overset{\wedge}{\phi}{}'_{\overline{\mu\nu}}\} = \frac{\delta_{xx'}}{J^2}\epsilon_{\mu\nu}{}^{\sigma\perp}\overset{\rightharpoonup}{\pi}_{\overline{\sigma}}, \tag{81}$$

$$\{\phi_{\perp}, \overset{\wedge}{\phi}{}'_{\overline{\mu\nu}}\} = \frac{\delta_{xx'}}{J^2}\overset{\wedge}{\pi}_{\overline{\mu\nu}\perp}, \tag{82}$$

$$\{{}^{\mathrm{T}}\phi_{\overline{\sigma\mu\nu}}, \overset{\wedge}{\phi}{}'_{\overline{\tau\rho}}\} = -\frac{\delta_{xx'}}{2J^2}{}^{\mathrm{T}}(\eta_{\overline{\mu}[\overline{\tau}}\eta_{\overline{\rho}]\overline{\nu}}\overset{\rightharpoonup}{\pi}_{\overline{\sigma}}), \tag{83}$$

$$\{\overset{\sim}{\phi}_{\overline{\mu\nu}\perp}, \overset{\wedge}{\phi}{}'_{\overline{\tau\sigma}}\} = \frac{\delta_{xx'}}{J^2}\overset{\wedge}{\pi}_{[\overline{\tau}\langle\overline{\mu}|\perp}\eta_{|\overline{\nu}\rangle\overline{\sigma}]}, \tag{84}$$

$$\{{}^{\mathrm{T}}\phi_{\overline{\sigma\mu\nu}}, \overset{\sim}{\phi}{}'_{\overline{\rho\tau}}\} = \text{rhs of Eq. (68)}, \tag{85}$$

$$\{\overset{\sim}{\phi}_{\overline{\mu\nu}\perp}, \overset{\sim}{\phi}{}'_{\overline{\tau\sigma}}\} = \text{rhs of Eq. (70)}. \tag{86}$$

By utilizing the PB's above the primary Poisson matrix has the form

$$
\begin{array}{ccc|cccccccc|}
 & & & \overbrace{\phi}^{1} & \overbrace{\phi_{\overline{\mu}\perp}}^{3} & \overbrace{\hat{\phi}_{\overline{\mu\nu}}}^{3} & \overbrace{\tilde{\phi}_{\overline{\mu\nu}}}^{5} & \overbrace{\phi_{\perp}}^{1} & \overbrace{{}^{\mathrm{P}}\phi}^{1} & \overbrace{\tilde{\phi}_{\overline{\mu\nu}\perp}}^{5} & \overbrace{{}^{\mathrm{T}}\phi_{\overline{\sigma\mu\nu}}}^{5} \\
1 & \{ & \phi & & & & & \eta & \bigcirc & \bigcirc & \bigcirc \\
3 & \{ & \phi_{\overline{\mu}\perp} & & & & & \pi & \pi & \pi & \pi \\
 & & & & & \bigcirc & & & & & \\
3 & \{ & \hat{\phi}_{\overline{\mu\nu}} & & & & & \pi & \pi & \pi & \pi \\
5 & \{ & \tilde{\phi}_{\overline{\mu\nu}} & & & & & \bigcirc & \bigcirc & \eta+\pi & \pi \\
1 & \{ & \phi_{\perp} & \eta & \pi & \pi & \bigcirc & & & & \\
1 & \{ & {}^{\mathrm{P}}\phi & \bigcirc & \pi & \pi & \bigcirc & & & & \\
 & & & & & & & & & \bigcirc & \\
5 & \{ & \tilde{\phi}_{\overline{\mu\nu}\perp} & \bigcirc & \pi & \pi & \eta+\pi & & & & \\
5 & \{ & {}^{\mathrm{T}}\phi_{\overline{\sigma\mu\nu}} & \bigcirc & \pi & \pi & \pi & & & &
\end{array}
\tag{87}
$$

The super-Hamiltonian density is

$$
\mathcal{H}_{\perp}^{1^-} = \mathcal{H}_{\perp}^{R1^+} + \mathcal{H}_{\perp}^{T1^-}, \tag{88}
$$

where

$$
\mathcal{H}_{\perp}^{T1^-} = \pi_{\overline{\mu}\perp}\left[\frac{3l^2}{2a_2}\frac{\pi^{\overline{\mu}\perp}}{J} + \overset{\rightharpoonup}{T}^{\overline{\mu}}\right] + \frac{l^2}{6a_2}\frac{\pi^2}{J} - n^{\mu}\nabla_a \pi^a{}_{\mu}. \tag{89}
$$

The constraints derived from the super-momenta and the Lorentz rotation parts are:

$$
\begin{aligned}
\mathcal{H}_{\overline{\mu}}^{1^-} \quad &\approx \quad R_{\overline{\mu\nu\sigma}\perp}\frac{\hat{\pi}^{\overline{\nu\sigma}\perp}}{J} - \frac{1}{2J}\underline{R}_{\overline{\mu\nu}}\overset{\rightharpoonup}{\pi}^{\overline{\nu}} \\
&\qquad - \frac{a_0}{l^2}R_{\overline{\mu}\perp} + \frac{\pi^{\overline{\nu}}{}_{\perp}}{J}\nabla_{\overline{\mu}}n_{\nu} - \frac{1}{3}\nabla_{\overline{\mu}}\frac{\pi}{J} \approx 0, \qquad (90) \\
\mathcal{H}_{\overline{\mu}\perp}^{1^+} \quad &\approx \quad \frac{\hat{\pi}_{\overline{\mu\nu}\perp}}{J}\overset{\rightharpoonup}{T}^{\overline{\nu}} - \frac{\pi_{\overline{\mu}\perp}}{J} - \frac{a_0}{l^2}\overset{\rightharpoonup}{T}_{\overline{\mu}} \\
&\qquad + \frac{1}{J}\overset{\rightharpoonup}{\pi}_{[\overline{\mu}}\nabla_{\overline{\nu}]}n^{\nu} - \delta_{\overline{\mu}}{}^{\sigma}\nabla_{\overline{\nu}}\frac{\hat{\pi}^{\overline{\nu}}{}_{\overline{\sigma}\perp}}{J} \approx 0, \qquad (91) \\
\mathcal{H}_{\overline{\mu\nu}}^{1^+} \quad &\approx \quad \frac{a_0}{2l^2}T_{\overline{\mu\nu}\perp} + \frac{1}{2J}\overset{\rightharpoonup}{\pi}_{[\overline{\mu}}\overset{\rightharpoonup}{T}_{\overline{\nu}]} \\
&\qquad + \frac{1}{2}\delta^{\sigma}{}_{[\overline{\mu}}\nabla_{\overline{\nu}]}\frac{\overset{\rightharpoonup}{\pi}_{\overline{\sigma}}}{J} - \frac{1}{J}\hat{\pi}^{\overline{\sigma}}{}_{[\overline{\mu}|\perp}\nabla_{\overline{\sigma}}n_{|\nu]} \approx 0. \qquad (92)
\end{aligned}
$$

By using same argument as in the simple spin-1^+ case, it is obvious that nonlinearity remains untamed. Linearly the degrees of freedom are $\frac{1}{2}(80-40-20[\mathrm{PIC}]-10[\mathrm{SC}]) = 5 = 3[1^-]+2[\mathrm{GR}]$. The spin-$1^-$ field is obtained as we expected. But nonlinearly it becomes $\frac{1}{2}(80-40-20[\mathrm{PIC}]-4[\mathrm{SC}]) = 8 = 3[1^-]+3[1^+]+2$. The appearance of spin-1^+ allows the possibility of negative kinetic energy and hinders the success of the case. The possible constraint bifurcation is noted and its consequence is similar to the spin-1^+ case discussed above.

In the aforementioned spin-one cases our analysis shows that neither the simple spin-1^+ mode nor the simple spin-1^- mode can be separated from each other. To see this more clearly, we can consider a wilder parameter arrangement: a_0 and b_5 non-vanishing only. This choice

has the consequence that all tetrad variables are not dynamical; they turn out to be the tetrad PIC's: ϕ, $\phi_{\overline{\mu}\perp}$, $\overset{\wedge}{\phi}_{\overline{\mu\nu}}$ and $\overset{\sim}{\phi}_{\overline{\mu\nu}}$. The other PIC's remain the same: ${}^{\mathrm{P}}\phi$, $\phi_{\perp}$, ${}^{\mathrm{T}}\phi_{\overline{\sigma\mu\nu}}$ and $\overset{\sim}{\phi}_{\overline{\mu\nu}\perp}$.

We find that generically the degrees of freedom count as $\frac{1}{2}(80-40-24[\mathrm{PIC}]) = 8 = 3[1^-]+3[1^+]+2$. The only difference between this "simpler" case and the simple spin-one cases discussed above is that the SC's induced from the PIC's in the simple spin-one cases have essentially transformed into PIC's coming from the parameter choice in this "simpler" case. However, the spin-one modes still stick to each other to crash the theory because of the negative kinetic energy.

5.1.3 The simple spin-2^- case

With the understanding of the simple spin-one cases the spin-2^+ mode which stands with the parameter choice of only a_0, a_1 and b_1 non-vanishing is not expected to have good physical propagation because it is entangled in a more complicated nonlinearity. Instead we examine the simple spin-2^- case. Although it eventually becomes futile, it is illuminating for our next case. According to Table 1 ${}^{\mathrm{T}}\pi_{\overline{\sigma\mu\nu}}$ corresponds to the spin-2^- mode. We make the specific parameter choices:

$$\begin{aligned} &a_0 \neq 0, \quad b_1 \neq 0,\\ &a_1 = a_2 = a_3 = 0,\\ &b_2 = b_3 = b_4 = b_5 = b_6 = 0. \end{aligned} \tag{93}$$

The PIC's are as follows:

$$\begin{aligned} \phi \equiv \frac{\pi}{J} \approx 0, &\qquad \overset{\sim}{\phi}_{\overline{\mu\nu}} \equiv \frac{\overset{\sim}{\pi}_{\overline{\mu\nu}}}{J} \approx 0,\\ \phi_{\overline{\mu}\perp} \equiv \frac{\pi_{\overline{\mu}\perp}}{J} \approx 0, &\qquad \overset{\wedge}{\phi}_{\overline{\mu\nu}} \equiv \frac{\overset{\wedge}{\pi}_{\overline{\mu\nu}}}{J} \approx 0,\\ {}^{\mathrm{P}}\phi \equiv \frac{{}^{\mathrm{P}}\pi}{J} \approx 0, &\qquad \overset{\rightharpoonup}{\phi}_{\overline{\mu}} \equiv \frac{\overset{\rightharpoonup}{\pi}_{\overline{\mu}}}{J} \approx 0,\\ \phi_{\perp} \equiv \frac{\pi}{J} - \frac{3a_0}{l^2} \approx 0, &\qquad \overset{\wedge}{\phi}_{\overline{\mu\nu}\perp} \equiv \frac{\overset{\wedge}{\pi}_{\overline{\mu\nu}\perp}}{J} \approx 0. \end{aligned} \tag{94}$$

The non-zero PB's are

$$\{{}^{\mathrm{P}}\phi, \overset{\sim}{\phi}{}'_{\overline{\mu\nu}}\} = \frac{4}{3}\frac{\delta_{xx'}}{J^2}\epsilon_{\langle\mu}{}^{\tau\sigma\perp}\,{}^{\mathrm{T}}\pi_{\overline{\nu}\rangle\overline{\tau\sigma}}, \tag{95}$$

$$\{\phi_{\perp}, \phi'\} = -\frac{\delta_{xx'}}{J}\frac{6a_0}{l^2}, \tag{96}$$

$$\{\phi_{\perp}, \overset{\sim}{\phi}{}'_{\overline{\mu\nu}}\} = \frac{\delta_{xx'}}{J^2}\,\overset{\sim}{\pi}_{\overline{\mu\nu}\perp}, \tag{97}$$

$$\{\overset{\rightharpoonup}{\phi}_{\overline{\mu}}, \phi'_{\overline{\nu}\perp}\} = \frac{\delta_{xx'}}{J}\left[\frac{1}{J}\overset{\sim}{\pi}_{\overline{\mu\nu}\perp} - \frac{2a_0}{l^2}\eta_{\overline{\mu\nu}}\right], \tag{98}$$

$$\{\overset{\rightharpoonup}{\phi}_{\overline{\mu}}, \overset{\wedge}{\phi}{}'_{\overline{\nu\sigma}}\} = \frac{2}{3}\frac{\delta_{xx'}}{J^2}\,{}^{\mathrm{T}}\pi_{\overline{\mu}[\overline{\nu\sigma}]}, \tag{99}$$

$$\{\overset{\rightharpoonup}{\phi}_{\overline{\mu}}, \overset{\sim}{\phi}{}'_{\overline{\nu\sigma}}\} = \frac{\delta_{xx'}}{J^2}\,{}^{\mathrm{T}}\pi_{\overline{\sigma\mu\nu}}, \tag{100}$$

$$\{\overset{\wedge}{\phi}_{\overline{\mu\nu}\perp}, \phi'_{\overline{\sigma}\perp}\} = -\frac{2}{3}\frac{\delta_{xx'}}{J^2}\,{}^{\mathrm{T}}\pi_{\overline{\sigma}[\overline{\mu\nu}]}, \tag{101}$$

$$\{\hat{\phi}_{\overline{\mu\nu}\perp}, \hat{\phi}'_{\overline{\tau\sigma}}\} = \frac{\delta_{xx'}}{J}\left[\frac{1}{J}\tilde{\pi}_{[\overline{\tau}[\overline{\nu}|\perp}\eta_{|\overline{\mu}]\overline{\sigma}]} + \frac{a_0}{l^2}\eta_{\overline{\sigma}[\overline{\mu}}\eta_{\overline{\nu}]\overline{\tau}}\right], \tag{102}$$

$$\{\hat{\phi}_{\overline{\mu\nu}\perp}, \tilde{\phi}'_{\overline{\tau\sigma}}\} = \frac{\delta_{xx'}}{J^2}\tilde{\pi}_{(\overline{\tau}[\overline{\nu}|\perp}\eta_{|\overline{\mu}]\overline{\sigma})}, \tag{103}$$

and the primary Poisson matrix has the form

$$\begin{array}{ccc|cccccccc|}
 & & & \overbrace{\phi}^{1} & \overbrace{\phi_{\overline{\mu}\perp}}^{3} & \overbrace{\hat{\phi}_{\overline{\mu\nu}}}^{3} & \overbrace{\tilde{\phi}_{\overline{\mu\nu}}}^{5} & \overbrace{\phi_{\perp}}^{1} & \overbrace{{}^{P}\phi}^{1} & \overbrace{\hat{\phi}_{\overline{\mu\nu}\perp}}^{3} & \overbrace{\vec{\phi}_{\overline{\mu}}}^{3} \\
1 & \{ & \phi & & & & & \eta & \bigcirc & \bigcirc & \bigcirc \\
3 & \{ & \phi_{\overline{\mu}\perp} & & & & & \bigcirc & \bigcirc & \pi & \eta+\pi \\
3 & \{ & \hat{\phi}_{\overline{\mu\nu}} & & \bigcirc & & & \bigcirc & \bigcirc & \eta+\pi & \pi \\
5 & \{ & \tilde{\phi}_{\overline{\mu\nu}} & & & & & \pi & \pi & \pi & \pi \\
1 & \{ & \phi_{\perp} & \eta & \bigcirc & \bigcirc & \pi & & & & \\
1 & \{ & {}^{P}\phi & \bigcirc & \bigcirc & \bigcirc & \pi & & & & \\
3 & \{ & \hat{\phi}_{\overline{\mu\nu}\perp} & \bigcirc & \pi & \eta+\pi & \pi & & & \bigcirc & \\
3 & \{ & \vec{\phi}_{\overline{\mu}} & \bigcirc & \eta+\pi & \pi & \pi & & & &
\end{array} \tag{104}$$

The corresponding super-Hamiltonian and the reduced constraints are

$$\begin{aligned}
\mathcal{H}_{\perp} &= \frac{\kappa}{3b_1}{}^{T}\pi_{\overline{\sigma\mu\nu}}\left(\frac{{}^{T}\pi^{\overline{\sigma\mu\nu}}}{J} + \frac{2b_1}{\kappa}{}^{T}R^{\overline{\mu\nu\sigma}}{}_{\perp}\right) - \frac{a_0}{2l^2}J\underline{R} \\
&\qquad + \frac{\kappa}{2b_1}\tilde{\pi}_{\overline{\mu\nu}\perp}\left(\frac{\tilde{\pi}^{\overline{\mu\nu}\perp}}{J} + \frac{2b_1}{\kappa}\underline{R}^{\langle\overline{\mu\nu}\rangle}\right) - n^{\mu}\nabla_a\pi^{a}{}_{\mu},
\end{aligned} \tag{105}$$

$$\mathcal{H}_a \longrightarrow \frac{2}{3}\frac{{}^{T}\pi_{\overline{\mu\nu\sigma}}}{J}\underline{R}^{[\overline{\nu\sigma}]} + \frac{{}^{T}\pi_{\overline{\sigma\mu\nu}}}{J}\underline{R}^{\langle\overline{\nu\sigma}\rangle} + \frac{\tilde{\pi}^{\overline{\nu\sigma}}{}_{\perp}}{J}R_{\overline{\mu\nu\sigma}\perp} + \frac{a_0}{l^2}R_{\overline{\mu}\perp} \approx 0, \tag{106}$$

$$\mathcal{H}_{\overline{\mu}\perp} \longrightarrow \frac{a_0}{l^2}\vec{T}_{\overline{\mu}} + \frac{\tilde{\pi}_{\overline{\mu\nu}\perp}}{J}\vec{T}^{\overline{\nu}} + \frac{4}{3}\frac{{}^{T}\pi_{[\overline{\mu\nu}]}{}^{\overline{\sigma}}}{J}\nabla_{\overline{\sigma}}n^{\nu} + \eta_{\overline{\mu\nu}}\nabla_{\overline{\sigma}}\frac{\tilde{\pi}^{\overline{\nu\sigma}\perp}}{J} \approx 0, \tag{107}$$

$$\begin{aligned}
\mathcal{H}_{\overline{\mu\nu}} &\longrightarrow \frac{4}{3}\left[\frac{{}^{T}\pi_{\overline{\sigma}[\overline{\mu\nu}]}}{J}\vec{T}^{\overline{\sigma}} + \eta_{\overline{\tau}[\overline{\mu}}\eta_{\overline{\nu}]\overline{\rho}}\nabla_{\overline{\sigma}}\frac{{}^{T}\pi^{\overline{\sigma\tau\rho}}}{J}\right] \\
&\qquad - \frac{2}{J}\tilde{\pi}^{\overline{\sigma}}{}_{[\overline{\mu}|\perp}\nabla_{\overline{\sigma}}n_{|\nu]} + \frac{a_0}{l^2}T_{\overline{\mu\nu}\perp} \approx 0.
\end{aligned} \tag{108}$$

Guided by our earlier experience, we readily realize that after its linearization the remaining non-zero PB's turn out to be Eq. (96,98,102) in which only the zero-order (constant) terms exist. Then the consistency conditions guarantee that the linearized Lagrange multipliers, i.e., $u^{\flat}$, $u^{\flat\perp}$, $u^{\flat\overline{\mu}\perp}$, $\vec{u}^{\flat\overline{\mu}}$, $\hat{u}^{\flat\overline{\mu\nu}}$ and $\hat{u}^{\flat\overline{\mu\nu}\perp}$, are obtained. The six PIC's, ${}^{P}\phi^{\flat}$ and $\tilde{\phi}^{\flat}_{\overline{\mu\nu}}$, and the six SC's, ${}^{P}\chi^{\flat}$ and $\tilde{\chi}^{\flat}_{\overline{\mu\nu}}$, which are derived from the consistency conditions of ${}^{P}\phi^{\flat}$ and $\tilde{\phi}^{\flat}_{\overline{\mu\nu}}$ are second-class pairs, respectively. These second-class pairs determine the multipliers ${}^{P}u^{\flat}$ and $\tilde{u}^{\flat\overline{\mu\nu}}$ and the algorithm of the linearized case is terminated. The degrees of freedom of the linearized case are counted as $\frac{1}{2}(80-40-20[\mathrm{PIC}]-6[\mathrm{SC}]) = 7 = 5[2^-]+2$, a massive spin-$2^-$ field in addition to the usual graviton.

On the other hand, the full spin-2^- case does not follow this track. Generically, $u^{\perp}$, $\vec{u}^{\overline{\mu}}$, $\hat{u}^{\overline{\mu\nu}\perp}$ can be determined. Since ${}^{P}\phi$ does not commute with $\tilde{\phi}_{\overline{\mu\nu}}$, the consistency condition of

$\widetilde{\phi}_{\overline{\mu\nu}}$ can determine ${}^{P}u$ and give four secondary constraints $\chi_{[4]}$. The consistency conditions of the connection primary constraints, $\phi_{\perp}$, ${}^{P}\phi$, $\hat{\phi}_{\overline{\mu\nu}}$ and $\vec{\phi}_{\overline{\mu}}$, leave four of the tetrad Lagrange multipliers, u, $u^{\overline{\mu}\perp}$, $\hat{u}^{\overline{\mu\nu}}$ and $\widetilde{u}^{\overline{\mu\nu}}$ undetermined. From the experience of the linearized case, we infer that the consistency condition of the four $\chi_{[4]}$ will allow the four undetermined tetrad Lagrange multipliers to be worked out and thereby terminate the process. The degrees of freedom would then count as $\frac{1}{2}(80-40-20[\text{PIC}]-4[\text{SC}]) = 8 = 5[2^-]+1+2$. It appears that one degree of freedom of the spin-2^+ mode is excited as well as the degrees of freedom that we expected. This is rather suspicious; a more careful analysis should be done to clarify these details. However, the main point is certain: once again the nonlinear effect makes the structure of the full simple spin-2^- case different from its linearized case.

The ill-behaviors in the cases above show that the existence of a single higher-spin mode is virtually impossible. Generally, in these cases there are extra degrees of freedom and the phenomenon of constraint bifurcation (compared with their linearized theories). Here the extra degrees of freedom represent ghosts, i.e., propagating negative energy, and the phenomenon of constraint bifurcation leads to tachyons, i.e., faster-than-light propagation. Therefore, nonlinearity makes the full theory qualitatively different from its linearization and the theory fails important theoretical tests.

The constraints coming from the reductions of the super-momenta and the Lorentz rotation parts do not seem to be much help in determining the existence of the extra degrees of freedom and the constraint bifurcation. This is because these constraints can be used to eliminate some unphysical canonical variables instead of unphysical canonical momenta. But only the canonical momenta are involved in the phenomena in these simple cases.

The reason we do not show the explicit forms of the SC's in these cases is because they are very complicated and we can obtain a sufficient understanding with only their linearized forms. The detailed form of these SC's is certainly needed in order to solve for the explicit form of the Lagrange multipliers. But the main point is that we know that this will be possible since they will not commute with all the primary constraints (in particular with their second-class pair counterparts). The complication of all the SC's in the full nonlinear theories comes from the entanglements of the variables and the momenta. These complications will not decrease the possibility of constraint bifurcation for the same reasons that were stated in the last paragraph.

5.2 Simple spin-0^- + spin-2^- cases

Although the inconsistency of the simple spin-2^- case with its linearization renders the theory unsuccessful, the sign of the few unexpected degrees of freedom appearing after including the nonlinear terms is notable. It becomes straightforward to recognize the situation from Table 1. In Table 1 only the PIC's ${}^{P}\phi$ and ${}^{T}\phi_{\overline{\sigma\mu\nu}}$ do not have their "natural" second-class pair counterparts coming from the tetrad PIC's; their second-class pair counterparts are the SC's derived from their own consistency conditions, i.e., ${}^{P}\dot{\phi}$ and ${}^{T}\dot{\phi}_{\overline{\sigma\mu\nu}}$. Thus the status of these two modes is unconcerned with the the tetrad parameter choice even in the full nonlinear case. This feature can be used advantageously; it will help in constructing some "more viable" simple cases. Here we show two parameter choices which give propagating ${}^{P}\pi$ and ${}^{T}\pi_{\overline{\sigma\mu\nu}}$ modes generically, albeit with drawbacks in the possible degeneracy.

5.2.1 The negative b_2 case

According to Table 1 the parameter b_2 appears in ${}^{\mathrm{P}}\phi(0^-)$, $\overset{\wedge}{\phi}_{\overline{\mu\nu}\perp}(1^+)$ and ${}^{\mathrm{T}}\phi_{\overline{\sigma\mu\nu}}(2^-)$. Here we make the tetrad parameters vanish in order to suppress the spin-1^+ mode. Therefore only the spin-0^- and spin-2^- modes propagate. the specific parameter choice is

$$\begin{aligned}
&a_0 \neq 0, \quad b_2 < 0, \\
&a_1 = a_2 = a_3 = 0, \\
&b_1 = b_3 = b_4 = b_5 = b_6 = 0.
\end{aligned} \tag{109}$$

This choice leads to the PIC's

$$\begin{aligned}
\phi \equiv \frac{\pi}{J} \approx 0, \qquad & \overset{\sim}{\phi}_{\overline{\mu\nu}} \equiv \frac{\overset{\sim}{\pi}_{\overline{\mu\nu}}}{J} \approx 0, \\
\phi_{\overline{\mu}\perp} \equiv \frac{\pi_{\overline{\mu}\perp}}{J} \approx 0, \qquad & \overset{\wedge}{\phi}_{\overline{\mu\nu}} \equiv \frac{\overset{\wedge}{\pi}_{\overline{\mu\nu}}}{J} \approx 0, \\
\phi_\perp \equiv \frac{\pi_\perp}{J} - \frac{3a_0}{l^2} \approx 0, \qquad & \overset{\rightharpoonup}{\phi}_{\overline{\mu}} \equiv \frac{\overset{\rightharpoonup}{\pi}_{\overline{\mu}}}{J} \approx 0, \\
\overset{\sim}{\phi}_{\overline{\mu\nu}\perp} \equiv \frac{\overset{\sim}{\pi}_{\overline{\mu\nu}\perp}}{J} \approx 0. \qquad &
\end{aligned} \tag{110}$$

The non-zero PB's are then

$$\{\phi_\perp, \phi'\} = -\frac{\delta_{xx'}}{J}\frac{6a_0}{l^2}, \tag{111}$$

$$\{\phi_\perp, \overset{\wedge}{\phi}{}'_{\overline{\mu\nu}}\} = \frac{\delta_{xx'}}{J^2}\,\overset{\wedge}{\pi}_{\overline{\mu\nu}\perp}, \tag{112}$$

$$\{\overset{\rightharpoonup}{\phi}_{\overline{\mu}}, \phi'_{\overline{\nu}\perp}\} = -\frac{\delta_{xx'}}{J}\left[\frac{1}{J}\overset{\wedge}{\pi}_{\overline{\mu\nu}\perp} + \frac{2a_0}{l^2}\eta_{\overline{\mu\nu}}\right], \tag{113}$$

$$\{\overset{\rightharpoonup}{\phi}_{\overline{\mu}}, \overset{\wedge}{\phi}{}'_{\overline{\nu\sigma}}\} = \frac{\delta_{xx'}}{J^2}\left[\frac{1}{6}{}^{\mathrm{P}}\pi\epsilon_{\mu\nu\sigma\perp} + \frac{2}{3}{}^{\mathrm{T}}\pi_{\overline{\mu}[\overline{\nu\sigma}]}\right], \tag{114}$$

$$\{\overset{\rightharpoonup}{\phi}_{\overline{\mu}}, \overset{\sim}{\phi}{}'_{\overline{\nu\sigma}}\} = \frac{\delta_{xx'}}{J^2}{}^{\mathrm{T}}\pi_{\overline{\sigma\mu\nu}}, \tag{115}$$

$$\{\overset{\sim}{\phi}_{\overline{\mu\nu}\perp}, \phi'_{\overline{\sigma}\perp}\} = -\frac{\delta_{xx'}}{J^2}{}^{\mathrm{T}}\pi_{\overline{\nu\sigma\mu}}, \tag{116}$$

$$\{\overset{\sim}{\phi}_{\overline{\mu\nu}\perp}, \overset{\wedge}{\phi}{}'_{\overline{\tau\sigma}}\} = \frac{\delta_{xx'}}{J^2}\overset{\wedge}{\pi}_{[\overline{\tau}\langle\overline{\mu}|\perp}\eta_{|\overline{\nu}\rangle\overline{\sigma}]}, \tag{117}$$

$$\{\overset{\sim}{\phi}_{\overline{\mu\nu}\perp}, \overset{\sim}{\phi}{}'_{\overline{\tau\sigma}}\} = \frac{\delta_{xx'}}{J}\left[\frac{a_0}{l^2}\eta_{\overline{\tau}\langle\overline{\mu}}\eta_{\overline{\nu}\rangle\overline{\sigma}} - \frac{1}{J}\eta_{(\overline{\tau}(\overline{\mu}}\overset{\wedge}{\pi}_{\overline{\nu})\overline{\sigma})\perp}\right]. \tag{118}$$

The primary Poisson matrix is

$$
\begin{array}{cc|ccccccc|}
 & & \overbrace{\phi}^{1} & \overbrace{\phi_{\overline{\mu}\perp}}^{3} & \overbrace{\hat{\phi}_{\overline{\mu\nu}}}^{3} & \overbrace{\tilde{\phi}_{\overline{\mu\nu}}}^{5} & \overbrace{\phi_{\perp}}^{1} & \overbrace{\vec{\phi}_{\overline{\mu}}}^{3} & \overbrace{\tilde{\phi}_{\overline{\mu\nu}\perp}}^{5} \\
1\ \{ & \phi & & & & & \eta & \bigcirc & \bigcirc \\
3\ \{ & \phi_{\overline{\mu}\perp} & & & & & \bigcirc & \eta+\pi & \pi \\
3\ \{ & \hat{\phi}_{\overline{\mu\nu}} & & & \bigcirc & & \pi & \pi & \pi \\
5\ \{ & \tilde{\phi}_{\overline{\mu\nu}} & & & & & \bigcirc & \pi & \eta+\pi \\
1\ \{ & \phi_{\perp} & \eta & \bigcirc & \pi & \bigcirc & & & \\
3\ \{ & \vec{\phi}_{\overline{\mu}} & \bigcirc & \eta+\pi & \pi & \pi & & \bigcirc & \\
3\ \{ & \tilde{\phi}_{\overline{\mu\nu}\perp} & \bigcirc & \pi & \pi & \eta+\pi & & &
\end{array}
\tag{119}
$$

The corresponding super-Hamiltonian and the reduced constraints are now

$$
\begin{aligned}
\mathcal{H}_{\perp} \quad = \quad & \frac{\kappa}{24b_2}{}^{\mathrm{P}}\pi\Big(\frac{{}^{\mathrm{P}}\pi}{J}-\frac{2b_2}{\kappa}{}^{\mathrm{P}}R_{\circ\perp}\Big)+\frac{\kappa}{3b_2}{}^{\mathrm{T}}\pi_{\overline{\sigma\mu\nu}}\Big(\frac{{}^{\mathrm{T}}\pi^{\overline{\sigma\mu\nu}}}{J}-\frac{2b_2}{\kappa}{}^{\mathrm{T}}R^{\overline{\mu\nu\sigma}}{}_{\perp}\Big) \\
& +\frac{\kappa}{2b_2}\hat{\pi}_{\overline{\mu\nu}\perp}\Big(\frac{\hat{\pi}^{\overline{\mu\nu}\perp}}{J}+\frac{2b_2}{\kappa}\underline{R}^{[\overline{\mu\nu}]}\Big)-\frac{a_0}{2l^2}J\underline{R}-n^{\mu}\nabla_a\pi^{a}{}_{\mu},
\end{aligned}
\tag{120}
$$

$$
\begin{aligned}
\mathcal{H}_{a} \quad \longrightarrow \quad & \left[\frac{1}{6}\frac{{}^{\mathrm{P}}\pi}{J}\epsilon_{\mu\nu\sigma\perp}+\frac{2}{3}\frac{{}^{\mathrm{T}}\pi_{\overline{\mu\nu\sigma}}}{J}\right]\underline{R}^{[\overline{\nu\sigma}]}+\frac{{}^{\mathrm{T}}\pi_{\overline{\sigma\mu\nu}}}{J}\underline{R}^{\langle\overline{\nu\sigma}\rangle} \\
& \qquad\qquad +\frac{\hat{\pi}^{\overline{\nu\sigma}}{}_{\perp}}{J}R_{\overline{\mu\nu\sigma}\perp}+\frac{a_0}{l^2}R_{\overline{\mu}\perp}\approx 0,
\end{aligned}
\tag{121}
$$

$$
\begin{aligned}
\mathcal{H}_{\overline{\mu}\perp} \quad \longrightarrow \quad & \frac{a_0}{2l^2}\vec{T}_{\overline{\mu}}-\frac{\hat{\pi}_{\overline{\mu\nu}\perp}}{J}\vec{T}^{\overline{\nu}}+\frac{1}{12}\frac{{}^{\mathrm{P}}\pi}{J}\epsilon_{\mu\nu\sigma\perp}T^{\overline{\nu\sigma}}{}_{\perp} \\
& \qquad\qquad +\frac{4}{3}\frac{{}^{\mathrm{T}}\pi_{[\overline{\mu\nu}]}{}^{\overline{\sigma}}}{J}\nabla_{\overline{\sigma}}n^{\nu}+\eta_{\overline{\mu\nu}}\nabla_{\overline{\sigma}}\frac{\hat{\pi}^{\overline{\nu\sigma}\perp}}{J}\approx 0,
\end{aligned}
\tag{122}
$$

$$
\begin{aligned}
\mathcal{H}_{\overline{\mu\nu}} \quad \longrightarrow \quad & \frac{4}{3}\left[\frac{{}^{\mathrm{T}}\pi_{\overline{\sigma}[\overline{\mu\nu}]}}{J}\vec{T}^{\overline{\sigma}}+\eta_{\overline{\tau}[\overline{\mu}}\eta_{\overline{\nu}]\overline{\rho}}\nabla_{\overline{\sigma}}\frac{{}^{\mathrm{T}}\pi^{\overline{\sigma\tau\rho}}}{J}\right]-\frac{2}{J}\hat{\pi}^{\overline{\sigma}}{}_{[\overline{\mu}|\perp}\nabla_{\overline{\sigma}}n_{|\nu]} \\
& \qquad -\frac{1}{6}\left[\frac{{}^{\mathrm{P}}\pi}{J}\epsilon_{\mu\nu\sigma\perp}\vec{T}^{\overline{\sigma}}+\frac{1}{6}\epsilon^{\sigma}{}_{\mu\nu\perp}\nabla_{\overline{\sigma}}\frac{{}^{\mathrm{P}}\pi}{J}\right]+\frac{a_0}{l^2}T_{\overline{\mu\nu}\perp}\approx 0.
\end{aligned}
\tag{123}
$$

Although this case seems more complicated, we found that the theory now has generically the same structure as its linearization. Briefly Eq. (111, 113, 118) are the only non-vanishing PB's after linearization because of their constant terms. Then the linearized multipliers $u^{\flat}$, $u^{\flat\perp}$, $u^{\flat\overline{\mu}\perp}$, $\vec{u}^{\flat\overline{\mu}}$, $\tilde{u}^{\flat\overline{\mu\nu}}$ and $\tilde{u}^{\flat\overline{\mu\nu}\perp}$ can be determined from the consistency conditions of the linearized PIC's $\phi^{\flat}$, $\phi^{\flat}_{\perp}$, $\phi^{\flat}_{\overline{\mu}\perp}$, $\vec{\phi}^{\flat}_{\overline{\mu}}$, $\tilde{\phi}^{\flat}_{\overline{\mu\nu}}$ and $\tilde{\phi}^{\flat}_{\overline{\mu\nu}\perp}$. Since $\hat{\phi}^{\flat}_{\overline{\mu\nu}}$ commutes with all the other linearized PIC's, its consistency condition gives the linearized SC's, $\hat{\chi}^{\flat}_{\overline{\mu\nu}}$, i.e.,

$$
\frac{\mathrm{d}}{\mathrm{d}t}\hat{\phi}^{\flat}_{\overline{\mu\nu}}\approx 0\longrightarrow\hat{\chi}^{\flat}_{\overline{\mu\nu}}=\frac{\hat{\pi}^{\flat}_{\overline{\mu\nu}\perp}}{J^{\flat}}-\frac{2b_2}{\kappa}\underline{R}^{\flat}_{[\overline{\mu\nu}]}\approx 0.
\tag{124}
$$

As expected $\hat{\pi}^{\flat}_{\overline{\mu\nu}\perp}$ appears in $\hat{\chi}^{\flat}_{\overline{\mu\nu}}$, so $\hat{\chi}^{\flat}_{\overline{\mu\nu}}$ and $\hat{\phi}^{\flat}_{\overline{\mu\nu}}$ are a second-class pair. Therefore the SC's can be used to determine the multiplier $\hat{u}^{\flat\overline{\mu\nu}}$. At the same time $\hat{\chi}^{\flat}_{\overline{\mu\nu}}$ also suppresses the the

spin-1^+ mode $\overset{\wedge}{\pi}{}^{\flat}_{\overline{\mu\nu}\perp}$ and prevents the system from having the problem of a propagating negative energy mode.

Generically the structure of the full theory is consistent with the one of the linearized theory despite the fact that the analysis is not as straightforward as the description in the last paragraph. The key difference is that $\overset{\wedge}{\phi}_{\overline{\mu\nu}}$ does not commute with the connection PIC's nonlinearly. The consistency condition,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\overset{\wedge}{\phi}_{\overline{\mu\nu}} &= \int\{\overset{\wedge}{\phi}_{\overline{\mu\nu}},\mathcal{H}'_{\mathrm{T}}\}\\ &\approx \int\Big[\{\overset{\wedge}{\phi}_{\overline{\mu\nu}},N'\mathcal{H}'_{\perp}\}+u^{\perp\prime}\{\overset{\wedge}{\phi}_{\overline{\mu\nu}},\phi'_{\perp}\}\\ &\qquad +\overset{\rightharpoonup}{u}{}^{\overline{\sigma}\prime}\{\overset{\wedge}{\phi}_{\overline{\mu\nu}},\overset{\rightharpoonup}{\phi}{}'_{\overline{\sigma}}\}+\tilde{u}{}^{\overline{\mu\nu}\perp\prime}\{\overset{\wedge}{\phi}_{\overline{\mu\nu}},\overset{\sim}{\phi}{}'_{\overline{\tau\sigma}\perp}\}\Big]\approx 0,\end{aligned}\tag{125}$$

can give the SC $\overset{\wedge}{\chi}_{\overline{\mu\nu}}$, here the connection multipliers shown in Eq. (125) can be derived from the consistency conditions of the other tetrad PICs. $\overset{\wedge}{\chi}_{\overline{\mu\nu}}$ includes $\overset{\wedge}{\chi}{}^{\flat}_{\overline{\mu\nu}}$ as the linear part and thus associates with $\overset{\wedge}{\phi}_{\overline{\mu\nu}}$ to become a second-class pair. Consequently, the SC renders $\overset{\wedge}{\phi}_{\overline{\mu\nu}\perp}$ non-dynamical. Now we have enough constraints to determine all the multipliers and finish the process. The degrees of freedom are $\frac{1}{2}(80-40-21[\mathrm{PIC}]-3[\mathrm{SC}])=8=1[0^-]+5[2^-]+2$ in both the linear and nonlinear theory. The result indicates that the theory can be transformed into its weak field approximation without qualitative changes in the canonical structure.

However the theory cannot entirely escape from the constraint bifurcation problem. By using `REDUCE` the possible vanishing of the determinant of the sub-Poisson matrix formed by the PB of $\phi_{\overline{\mu}\perp}$, $\overset{\sim}{\phi}_{\overline{\mu\nu}}$, $\overset{\rightharpoonup}{\phi}_{\overline{\mu}}$ and $\overset{\sim}{\phi}_{\overline{\mu\nu}\perp}$ is affirmed, i.e., with certain numerical values of the canonical variables which satisfy all constraints the sub-Poisson matrix can be singular. This is not so good. Such an occurrence is thought to be a sign of the presence of a mode which propagates faster than light[41].

5.2.2 The b_1+b_3 case

The other parameter choice available to activate only the spin-0^- and spin-2^- modes is to let all parameters vanish except a_0, b_1 and b_3. This choice gives the PIC's ϕ, $\phi_{\overline{\mu}\perp}$, $\overset{\wedge}{\phi}_{\overline{\mu\nu}}$, $\overset{\sim}{\phi}_{\overline{\mu\nu}}$, $\phi_{\perp}$, $\overset{\wedge}{\phi}_{\overline{\mu\nu}\perp}$ and $\overset{\rightharpoonup}{\phi}_{\overline{\mu}}$. The non-zero PB's are

$$\{\phi_{\perp},\phi'\} = -\frac{\delta_{xx'}}{J}\frac{6a_0}{l^2},\tag{126}$$

$$\{\phi_{\perp},\overset{\sim}{\phi}{}'_{\overline{\mu\nu}}\} = \frac{\delta_{xx'}}{J^2}\overset{\sim}{\pi}_{\overline{\mu\nu}\perp},\tag{127}$$

$$\{\overset{\wedge}{\phi}_{\overline{\mu\nu}\perp},\phi'_{\overline{\sigma}\perp}\} = \frac{\delta_{xx'}}{J^2}\left[\frac{2}{3}{}^{\mathrm{T}}\pi_{[\overline{\mu\nu}]\overline{\sigma}}-\frac{1}{6}{}^{\mathrm{P}}\pi\epsilon_{\mu\nu\sigma\perp}\right],\tag{128}$$

$$\{\overset{\wedge}{\phi}_{\overline{\mu\nu}\perp},\overset{\wedge}{\phi}{}'_{\overline{\tau\sigma}}\} = \frac{\delta_{xx'}}{J}\left[\frac{a_0}{l^2}\eta_{\overline{\sigma}[\overline{\mu}}\eta_{\overline{\nu}]\overline{\tau}}+\frac{1}{J}\overset{\sim}{\pi}_{[\overline{\sigma}[\overline{\mu}|\perp}\eta_{|\overline{\nu}]\overline{\tau}]}\right],\tag{129}$$

$$\{\overset{\wedge}{\phi}_{\overline{\mu\nu}\perp},\overset{\sim}{\phi}{}'_{\overline{\tau\sigma}}\} = \frac{\delta_{xx'}}{J^2}\eta_{[\overline{\mu}(\overline{\sigma}}\overset{\sim}{\pi}_{\overline{\tau})\overline{\nu}]\perp},\tag{130}$$

$$\{\overset{\rightharpoonup}{\phi}_{\overline{\mu}},\phi'_{\overline{\nu}\perp}\} = \frac{\delta_{xx'}}{J}\left[\frac{1}{J}\overset{\sim}{\pi}_{\overline{\mu\nu}\perp}-\frac{2a_0}{l^2}\eta_{\overline{\mu\nu}}\right],\tag{131}$$

$$\{\vec{\phi}_{\bar{\mu}}, \overset{\wedge}{\phi}'_{\overline{\nu\sigma}}\} = \frac{\delta_{xx'}}{J^2}\left[\frac{1}{6}{}^{\mathrm{P}}\pi\epsilon_{\mu\nu\sigma\perp} + \frac{2}{3}{}^{\mathrm{T}}\pi_{\bar{\mu}[\overline{\nu\sigma}]}\right], \tag{132}$$

$$\{\vec{\phi}_{\bar{\mu}}, \tilde{\phi}'_{\overline{\nu\sigma}}\} = \frac{\delta_{xx'}}{J^2}{}^{\mathrm{T}}\pi_{\overline{\sigma\mu\nu}}. \tag{133}$$

and the primary Poisson matrix is

$$\begin{array}{lll|ccccccc|}
 & & & \overbrace{\phi}^{1} & \overbrace{\phi_{\bar{\mu}\perp}}^{3} & \overbrace{\overset{\wedge}{\phi}_{\overline{\mu\nu}}}^{3} & \overbrace{\tilde{\phi}_{\overline{\mu\nu}}}^{5} & \overbrace{\phi_{\perp}}^{1} & \overbrace{\overset{\wedge}{\phi}_{\overline{\mu\nu}\perp}}^{3} & \overbrace{\vec{\phi}_{\bar{\mu}}}^{3} \\
1 & \{ & \phi & & & & & \eta & \bigcirc & \bigcirc \\
3 & \{ & \phi_{\bar{\mu}\perp} & & & & & \bigcirc & \pi & \eta+\pi \\
3 & \{ & \overset{\wedge}{\phi}_{\overline{\mu\nu}} & & & \bigcirc & & \bigcirc & \eta+\pi & \pi \\
5 & \{ & \tilde{\phi}_{\overline{\mu\nu}} & & & & & \pi & \pi & \pi \\
1 & \{ & \phi_{\perp} & \eta & \bigcirc & \bigcirc & \pi & & & \\
3 & \{ & \overset{\wedge}{\phi}_{\overline{\mu\nu}\perp} & \bigcirc & \pi & \eta+\pi & \pi & & \bigcirc & \\
3 & \{ & \vec{\phi}_{\bar{\mu}} & \bigcirc & \eta+\pi & \pi & \pi & & &
\end{array} \tag{134}$$

The corresponding super-Hamiltonian and the reduced constraints are

$$\begin{aligned}
\mathcal{H}_{\perp} = {} & \frac{\kappa}{24b_3}{}^{\mathrm{P}}\pi\left(\frac{{}^{\mathrm{P}}\pi}{J} + \frac{2b_3}{\kappa}{}^{\mathrm{P}}R_{\circ\perp}\right) + \frac{\kappa}{3b_1}{}^{\mathrm{T}}\pi_{\sigma\mu\nu}\left(\frac{{}^{\mathrm{T}}\pi^{\overline{\sigma\mu\nu}}}{J} + \frac{2b_1}{\kappa}{}^{\mathrm{T}}R^{\overline{\mu\nu\sigma}}{}_{\perp}\right) \\
& + \frac{\kappa}{2b_1}\tilde{\pi}_{\overline{\mu\nu}\perp}\left(\frac{\tilde{\pi}^{\overline{\mu\nu}\perp}}{J} + \frac{2b_1}{\kappa}\underline{R}^{\langle\overline{\mu\nu}\rangle}\right) - \frac{a_0}{2l^2}J\underline{R} - n^{\mu}\nabla_a\pi^a{}_{\mu},
\end{aligned} \tag{135}$$

$$\begin{aligned}
\mathcal{H}_{a} \longrightarrow {} & \left[\frac{1}{6}\frac{{}^{\mathrm{P}}\pi}{J}\epsilon_{\mu\nu\sigma\perp} + \frac{2}{3}\frac{{}^{\mathrm{T}}\pi_{\overline{\mu\nu\sigma}}}{J}\right]\underline{R}^{[\overline{\nu\sigma}]} + \frac{{}^{\mathrm{T}}\pi_{\overline{\sigma\mu\nu}}}{J}\underline{R}^{\langle\overline{\nu\sigma}\rangle} \\
& + \frac{\tilde{\pi}^{\overline{\nu\sigma}}{}_{\perp}}{J}R_{\overline{\mu\nu\sigma}\perp} + \frac{a_0}{l^2}R_{\bar{\mu}\perp} \approx 0,
\end{aligned} \tag{136}$$

$$\begin{aligned}
\mathcal{H}_{\bar{\mu}\perp} \longrightarrow {} & \frac{a_0}{l^2}\vec{T}_{\bar{\mu}} + \frac{\tilde{\pi}_{\overline{\mu\nu}\perp}}{J}\vec{T}^{\bar{\nu}} + \frac{1}{12}\frac{{}^{\mathrm{P}}\pi}{J}\epsilon_{\mu\nu\sigma\perp}T^{\overline{\nu\sigma}}{}_{\perp} \\
& + \frac{4}{3}\frac{{}^{\mathrm{T}}\pi_{[\overline{\mu\nu}]}{}^{\bar{\sigma}}}{J}\nabla_{\bar{\sigma}}n^{\nu} + \eta_{\overline{\mu\nu}}\nabla_{\bar{\sigma}}\frac{\tilde{\pi}^{\overline{\nu\sigma}\perp}}{J} \approx 0,
\end{aligned} \tag{137}$$

$$\begin{aligned}
\mathcal{H}_{\overline{\mu\nu}} \longrightarrow {} & \frac{4}{3}\left[\frac{{}^{\mathrm{T}}\pi_{\bar{\sigma}[\overline{\mu\nu}]}}{J}\vec{T}^{\bar{\sigma}} + \eta_{\bar{\tau}[\bar{\mu}}\eta_{\bar{\nu}]\bar{\rho}}\nabla_{\bar{\sigma}}\frac{{}^{\mathrm{T}}\pi^{\overline{\sigma\tau\rho}}}{J}\right] - \frac{2}{J}\tilde{\pi}^{\bar{\sigma}}{}_{[\bar{\mu}|\perp}\nabla_{\bar{\sigma}}n_{|\nu]} \\
& - \frac{1}{6}\epsilon^{\sigma}{}_{\mu\nu\perp}\left[\frac{{}^{\mathrm{P}}\pi}{J}\vec{T}_{\bar{\sigma}} + \nabla_{\bar{\sigma}}\frac{{}^{\mathrm{P}}\pi}{J}\right] + \frac{a_0}{l^2}T_{\overline{\mu\nu}\perp} \approx 0.
\end{aligned} \tag{138}$$

Essentially the same argument as used in the negative b_2 case can be applied to this case. The degrees of freedom are generically $\frac{1}{2}(80 - 40 - 19[\mathrm{PIC}] - 5[\mathrm{SC}]) = 8 = 1[0^-] + 5[2^-] + 2$ in both the linear and nonlinear theory. But once again it is readily seen that this case also almost certainly has constraint bifurcation and consequently is very vulnerable to acausal propagation.

Two questions are raised by these results: (a) Could relaxed versions of the two spin-0^- + 2^- modes overcome the defect of constraint bifurcation? (b) Do there exist any parameter

choices which make the compound propagating modes nonlinearly viable? Concerning the first question, according to the explanation in the beginning of this chapter, more nonlinear effects will be involved in the relaxed versions. The Poisson matrix is supposed to be more complicated and even include the spatial derivatives of $\delta_{xx'}$. It will become harder to analyze the whole system and to convince people that it has the advantage of being free of any constraint bifurcation problems. To the second question, basically this is beyond our ability to give a definite answer. Maybe with some delicate tuning of all parameters there exist such nonlinearly viable cases of the PGT. (After all, unexpected miraculous cancellations have been found in other systems.) However, based on our experience with the PGT, we hold a very conservative attitude to the possibility of this happening.

In order to convince ourselves that the cases dubbed "simple" in this section do show the obstacles that all the relaxed cases and many similar cases could meet in the Hamiltonian analysis, it seems worthwhile to investigate the most promising parameter choice coming from the linearized PGT. In section 4 the parameter choice (59) given by Kuhfuss and Nitsch[1] was quite restrictive and supposed to be "viable". Here we give a brief analysis of the theory with this parameter choice. Referring to Table 1 we learn that the PIC's in the theory are $\phi_{\overline{\mu}\perp}$, $\overset{\wedge}{\phi}_{\overline{\mu\nu}}$, $\phi_{\perp}$, $\overset{\wedge}{\phi}_{\overline{\mu\nu}\perp}$ and $\overset{\sim}{\phi}_{\overline{\mu\nu}\perp}$. Under linearization one will expect that $\overset{\wedge}{\phi}{}^{\flat}_{\overline{\mu\nu}}$ and $\overset{\wedge}{\phi}{}^{\flat}_{\overline{\mu\nu}\perp}$ are first-class due to $a_1 = a_0$. And $\phi^{\flat}_{\overline{\mu}\perp}$, $\phi^{\flat}_{\perp}$ and $\overset{\sim}{\phi}{}^{\flat}_{\overline{\mu\nu}\perp}$ commute with one another. Subsequently $\chi^{\flat}_{\overline{\mu}\perp}$, $\chi^{\flat}_{\perp}$ and $\tilde{\chi}^{\flat}_{\overline{\mu\nu}\perp}$ are derived from their parent constraints. These SC's and their parent constraints form second-class pairs and thus suppress $\pi^{\flat}_{\overline{\mu}\perp}$, $\pi^{\flat}_{\perp}$ and $\tilde{\pi}^{\flat}_{\overline{\mu\nu}\perp}$ respectively. The degrees of freedom count as $\frac{1}{2}(80 - 40 - 15[\mathrm{PIC}] - 9[\mathrm{SC}] - 6[\mathrm{gauges}]) = 5 = 2[\mathrm{GR}] + 3$. It is easy to discover that the three degrees of freedom come from the massless spin-0^- and spin-2^- propagating modes.

Now we return to the full nonlinear case. In order to make the analysis simpler, we can specify the parameters to be $b_1 = b_4 = b_6 = 0$. First, $\phi_{\overline{\mu}\perp}$ and $\overset{\wedge}{\phi}_{\overline{\mu\nu}\perp}$ will not commute with the connection PIC's because of the nonlinear terms. Consequently we expect that the consistency conditions of all the PIC's will determine twelve Lagrange multipliers and give three SC's $\chi_{[3]}$. Using $\chi_{[3]}$, the remaining three multipliers can be obtained. The degrees of freedom would then count as $\frac{1}{2}(80-40-15[\mathrm{PIC}]-3[\mathrm{SC}]) = 11 = 2[\mathrm{GR}]+1[0^-]+3[1^-]+5[2^-]$; all excited propagating modes have become "massive" formally. Nonlinear effects have excited extra dynamic degrees of freedom and destroyed the gauge freedom of the linearized theory. Moreover, if we require the kinetic energy density of the spin-2^- mode to be positive definite, it is inevitable that the spin-1^- mode propagates with negative energy density. This unexpected mode with the wrong sign shows another defect of this parameter choice.

Here we try to illustrate that the relaxed cases can be even more vulnerable to getting stuck in troubles. Essentially, the Kuhfuss & Nitsch's case is in the same track as ours in Section 5.2. They concluded that only the spin-0^- and spin-2^- modes could propagate. We found from the nonlinear structure of the PGT that it is usually unlikely to suppress one single higher-spin ($s > 0$) mode without suppressing its counterpart if they form a second-class pair in the linearized theory. This realization forces the parameter choice to be the one which only activates the modes not belonging to any second-class pairs — the spin-0^- and spin-2^- modes.

Maybe there exists still the thought that it might still work with different values of the nonzero parameters or in more relaxed versions. Until we do further more detailed investigations this possibility cannot be dismissed. Along this line one has to face more problems: (1) Because the connection PIC's do not commute generally in the relaxed cases, more non-zero PB's appear, which could excite extra degrees of freedom. Negative energy can be expected to appear in

most of those cases; (2) In general, the spatial derivatives of the $\delta_{xx'}$ function could appear in the PB's of the connection PIC's. This greatly complicates the difficulty of analyzing the generic rank of the Poisson matrix. Even if these problems are settled down well there is the very possible phenomenon of constraint bifurcation with its attendant acausal modes waiting ahead!

6 Discussion and Conclusion

In this paper we examined the behavior of a restricted case of the MAG — the PGT, the local gauge theory of the Poincaré group. People used to work on the linearized PGT and its initial value problems to clarify its viability, however they overlooked the significant influences of nonlinearity on the whole theory. In Sec. 3 we noted that two problems are produced by nonlinearity: (1) *Constraint bifurcation*: the phenomenon is caused by the appearances of the nonlinear terms in the constraints which lead to a field dependence in the Poisson matrix. This can cause a bifurcation in the constraint chain. The number and type of constraints can depend on the field values. This phenomenon has been linked to acausal (tachyonic) propagation modes[40,41]. (2) *Field activation*: the phenomenon is that nonlinearity turns some original constraints in the linearized theories into field equations. This means that some field modes which should be frozen are activated. If the fields carry negative energy, it violates the "no-ghost" requirement. We regard these two phenomena as very important; our criteria is that they should be avoided. A good theory should keep the same dynamical structure before and after linearization; it should not be bothered by these two phenomena. We find it hard to imagine the consequences of a theory of a fundamental interaction which would exhibit a different dynamical structure as we passed from the strong field to weak field regime.

In the last section we examined nonlinear effects in the PGT which can switch the status of presumably viable parameter choices. The main tools are the *Dirac-Bergmann algorithm* of Hamiltonian analysis, and the *if-constraint* technique developed by Blagojević and Nikolić (described in Sec. 3). The procedures we used are as follows:

(a) Make the desired parameter choice according to the study of the linearized PGT.

(b) Identify all the constraints in the selected case and calculate their Poisson brackets.

(c) Classify the constraints by the results of the Poisson brackets. From the consistency conditions find out all secondary constraints and the Lagrange multipliers.

(d) Count the degrees of freedom of the selected case. By knowing the corresponding linearized results, we can figure out the meaning of each degree of freedom without further calculation.

(e) Evaluate whether the Poisson matrix formed by the Poisson brackets of all the constraints can be singular at specific values of the variables in order to determine any occurrence of constraint bifurcation. The computer algebraic software `REDUCE` can be applied to work out the lengthy calculations.

Individual "simple" spin modes, which are well behaved and thus viable in the linearized PGT, were chosen to study their behavior under the full nonlinear considerations. For the spin zero modes, we previously found that they are essentially viable although nonlinear effects

obviously complicate the whole systems We noted that there is the possibility of constraint bifurcation, however, this can be avoided with specific value choices of the parameters[21]. Here we concluded that the nonlinear terms simply devastate the viabilities of the spin one modes and spin two modes, since each expectedly frozen mode which has the same spin but opposite parity with the propagating mode in every case is activated by nonlinearity. Unfortunately, the nonlinear-activated mode propagates with negative energy which is physically unacceptable theoretically.

There can be maximally three different propagating torsion modes in the linearized PGT as described in Sec. 4. Therefore we looked for the possibly viable multi-modes according to the classification in Ref. 7. Through many trial-and-errors only two "simple" cases of multi-modes which circumvent the difficulty of the activation of the unexpected modes are found, i.e., keep only the spin-0^- and spin-2^- modes alive and suppress all the other spin modes. Successfully the structure (and the degrees of freedom) of the full nonlinear theory remains the same as that of its linearization in these two case. This is a promising results. Even though we argued that the constraint bifurcation of the Poisson matrices could be expected to render them unattractive, we wish to note that that issue, and these modes more generally, are worthy of a deeper study.

There are a few other special parameter choices meriting examiniation, e.g., those identified by Katanaev which give solutions with vanishing curvature or torsion[10]. One that we specifically considered here was Kuhfuss and Nitsch's case, which was thought to be the most promising case for being viable. We found that the case changes its character drastically between the nonlinear and linearized versions. Both the phenomena of constraint bifurcation and field activation happen; they devastate the case. Considering this result, one could be pessimistic about finding viable higher spin modes in the nonlinear PGT. But we don't exclude the possibility, with a fine tuning of all the involved parameters. Once the case exists, the propagating modes are expected to be short-range, i.e., massive, just like the viable spin zero modes.

It seems that only a few cases in the PGT can avoid the phenomenon of field activation. And almost all of these cases are plagued with the constraint bifurcation problem. This means that the nonlinear PGT is very likely to be qualitatively different from the linearized PGT in the number and type of constraints. The more general MAG theory includes the PGT as a subcase and thus already has these same problems. Moreover it seems extremely likely that nonmetricity itself will be vulnerable to the same kind of problems.

Our analysis of non-linear constraint effects in the PGT reveals an extreme difficulty in finding a viable gauge alternative of gravity which had not been appreciated in the past. In the light of this, a further important consequence of our analysis is an enhanced understanding of the advantages and uniqueness of GR.

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